Collaborative Search for a Public Good[*]

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April 2021

LATEST VERSION

Abstract

This paper studies a model of costly sequential search among risky alternatives performed by a group of agents. The learning process stops, and the best uncovered option is implemented when the agents unanimously agree to stop or when all the projects have been researched. Both the implemented project and all the information gathered during the search process are public goods. I show that the equilibrium path implements the same project based on the same information gathered in the same order as the social planner. At the same time, due to free riding, search in teams leads to a delay at each stage of the learning process, which grows with search costs. Consequently, the team manager prefers to delegate the search to an individual agent. In contrast, every agent prefers searching with a partner, since she collects the same reward but only pays the search cost half the time.

KEYWORDS: Information Acquisition, Optimal Search, Committees

JEL CLASSIFICATION: D71, D83

[*]I am particularly grateful to my friend and colleague Alex Weiss for productive conversations, support, and encouragement throughout the course of writing this paper. I thank my advisors Renee Bowen and Joel Sobel for useful discussions and detailed feedback. I am also grateful to the seminar participants at UC San Diego for their comments and feedback. All errors are my own.

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1. INTRODUCTION

Suppose a team of agents is facing a problem, solving which benefits them all. Before making a collective choice, the team members must engage in a costly search to learn the possible solutions. How efficient is collaborative search?

Collaborative search for a public good takes place in many economic situations. In politics, policymakers identify the best available policies. In organizations, committee members search for the most qualified candidate. In consumer search, family members look for a house to move to. In research and development, scientists decide which idea to pursue. Broadly speaking, any situation that involves sequential social learning and that results in the final project benefiting everyone can be studied using this model. I examine inefficiencies that arise as an artifact of sequential searching in teams rather than individually.

I model the sequential search process after the seminal model of Weitzman (1979). There are two team members and a finite number of boxes. Each box contains an uncertain reward. To learn the contents of a box, one needs to open it, which comes at a cost. At each stage of the game, one agent is randomly chosen to decide between three alternatives: she could open a box of her choice, do nothing, or propose to terminate the game. The game ends if a termination offer is extended and accepted or if there are no more boxes left to open. At the end of the game, both players collect the highest reward among all the opened boxes.[1] I study *(i)* the optimal order of search among alternatives, *(ii)* incentives to free ride on colleague's search efforts, *(iii)* the efficiency of searching in teams.

My most important result is that, compared to the socially optimal protocol of an individual searcher, the team will use the same search order and stopping rule. In other words, the policymakers identify the same policy, the committee members find the same candidate, the family moves to the same house, and the scientists pursue the same research project as if these choices were made by the social planner. However, team search may be inefficient due to the free riding effect: agents procrastinate at each stage of the search

[1] Weitzman's setup was generalized by Olszewski and Weber (2015) to have the searcher's payoff depends on all discovered prizes, and by Doval (2018) to allow the searcher to forego inspection costs and take any unopened box. I focus on Weitzman's setup because of the simple form of the solution that allows for a straightforward comparison of the team vs. social planner results.

process and hope their colleagues exert the search effort instead.

In the symmetric equilibrium, the chosen player acts similarly to how she would act had she been searching alone. More precisely, if she decides to open a box, she opens the "best" box according to Weitzman (1979), i.e. the box with the highest reservation value. She also wants to stop and proposes to terminate the game at the same threshold – when no boxes are "good enough" to be opened, i.e. the highest reservation value among the unopened boxes is lower than best uncovered reward so far. In that case, her opponent always accepts the termination offer. Consequently, the search order and termination protocol on the equilibrium path are those of the social planner. The only difference is that agents free ride when search costs are sufficiently high: the chosen player only opens a box sometimes, and does nothing the rest of the time. As a result, delay arises at each stage of the learning process.

Compared to searching by herself, each agent is doing better on average when searching with a companion. Intuitively, their learning protocol is the same, but each team member only pays the search cost about half the time. However, team search is inefficient because every box is opened with a delay. Delay occurs at each phase of the learning process in the sense that it takes time to open each consecutive box. How long it takes to open a box depends on the distribution of the reward and the best uncovered option so far. As players open boxes, two effects take place. First, because the search protocol prescribes to open ex-ante better boxes first, ex-ante worse boxes remain towards the end. Hence, the next best box to be opened is less attractive than the previous box and takes longer to open. Second, when someone opens a box, the best uncovered option improves. That decreases the time it takes to open the following box because the players are eager to collect the higher reward. As a result, the time it takes to open the next box may go up or down.

RELATED LITERATURE

First, this paper contributes to the literature on collective experimentation. In their seminar paper, Bolton and Harris (1999) extend the classic two-armed bandit problem to a many-agent setting. Because information is a public good, agents have an incentive to wait and let their colleagues experiment instead, which is known as the *free rider effect*. At the same time, the prospect of others experimenting forces every agent to experiment more, which is known as the *encouragement effect*. The strength of each of these effects depends on the problem. For example, if the bandit is exponential, as in Keller, Rady, and Cripps (2005), then only the free-riding effect is present. With Poisson bandits, Keller and Rady (2010) show that the encouragement effect dominates. All these papers focus on a two-armed bandit that has one safe, one risky arm. In this paper, I consider a multi-armed bandit, such that the outcome of an arm is revealed after just one experiment. This allows me to study the *order* as well as the *stopping rule* of the search process for the public good.

The literature on collective experimentation is closely related to the literature on delegation and approval of experimentation. When a principal delegates experimentation to agents, the optimal mechanism exhibits tolerance for early failure (Manso, 2011; Lewis, 2012), but asymmetric information leads to less experimentation, lower success rate, and more variance in success rates (Halac, Kartik, and Liu, 2016). When the agent is privately informed about the state of the world, optimal dynamic mechanisms feature cutoff approval rules (Guo, 2016; McClellan, 2019). I abstract away from the principal's incentive-compatibility problem and study individual and team incentives instead. I conclude that from the outside (i.e. the manager's) perspective, search by an individual is more efficient because it happens without any delay. At the same time, each agent would rather search with a companion because that allows her to pay the search cost less often.

This paper contributes to the literature on the search by committees. Group experimentation that ends with a vote is usually inefficient because committee members experience loser trap and winner frustration (Strulovici, 2010), are less picky and more conservative than a single agent (Albrecht, Anderson, and Vroman, 2010), or because they communicate before the game (Compte and Jehiel, 2010). In my setting, the agents are collectively searching for the public good. Since their preferences are aligned, agents agree to end the learning process when the social planner would, which is true for any social choice function.

This paper also contributes to the literature on collaboration in teams. When agents work on a project as a team, inefficiencies arise because team members have an incentive to procrastinate (Bonatti and Hörner, 2011) and due to lack of communication (Campbell, Ederer, and Spinnewijn, 2014). The size of the inefficiency is minimized if the manager dynamically decreases the size of the team as the project nears completion (Georgiadis, 2015). My findings confirm that agents free ride when searching in teams. At the same time, they prefer to search as part of a team because when they procrastinate, there is a chance that their partner exerts the costly search effort.

Finally, this paper contributes to the literature on the dynamic provision of public goods. Efficiency is usually not achieved because socially optimal projects are not completed (Fershtman and Nitzan, 1991; Admati and Perry, 1991; Kessing, 2007), completed with a delay (Marx and Matthews, 2000; Compte and Jehiel, 2004), or completed at a lower scale (Bowen, Georgiadis, and Lambert, 2019). I show that when searching for the public good, team members effectively use the socially optimal search protocol, albeit with some free riding when search costs are large enough. Consequently, all socially optimal projects are searched through, and the only source of inefficiency is the delay.

The rest of the paper is organized as follows. Section 2 introduces the dynamic model of sequential search among risky alternatives. Section 3 describes the model with one alternative and discusses comparative statics and welfare implications. Section 4 generalizes the model to the case of finitely many alternatives. Section 5 is a conclusion.

2. Model

Two agents sequentially search for a public good. At each stage, one agent is chosen randomly with a probability of $1/2$. The chosen player has an option to *(i)* open exactly one box of her choice, *(ii)* do nothing, or *(iii)* propose to terminate the game. In the latter case, her opponent chooses between accepting and rejecting this proposal.

Each public good project is represented by a box that contains a stochastic prize. Initially, there is a finite number of unopened boxes. Box $b_k = (c_k, F_k)$ contains an uncertain reward $x_k \sim F_k(\cdot)$ distributed independently of all other rewards. If the chosen player decides to open this box, she pays the search cost c_k and players wait one period to learn its contents. Once the contents are revealed, a new stage starts immediately, and a new player is chosen. The game ends if the chosen player proposes termination and the opponent accepts the offer, or if there are no more boxes left to open. In either case, each player collects the highest reward they uncovered during the search. The initial fallback reward is z_0.

Both players are risk-neutral and wish to maximize the expected present value of the best uncovered reward. The search costs are sunk because they are paid during the search process, while the reward is only realized upon the end of the game. The players discount the time at the exponential rate $\delta = e^{-r\Delta t}$, where Δt is the length of the time interval between the stages. This paper aims to find a dynamic rule that describes the optimal search protocol. This dynamic rule should specify which box (if any) to open, when to propose to end the search process, and when to accept the termination offer.

THE DYNAMIC PROBLEM

Let $\mathcal{B}$ denote the set of unopened boxes and z be the best uncovered reward. I focus on the stationary Markov perfect equilibrium. It is a subgame perfect equilibrium in which strategies depend only on the payoff-relevant history, i.e. the pair $(z, \mathcal{B})$.

A stationary Markov strategy for player i is a pair $a_i := (a_i^{ch}, a_i^{op})$ that specifies which action she takes when she is chosen and when she is the opponent, respectively. With slight abuse of notation, $a_i^{ch}(z, \mathcal{B}) \in A^{ch}(z, \mathcal{B}) := \{\varnothing, T\} \cup \mathcal{B}$, meaning that when chosen, player i decides between doing nothing, proposing termination, and opening one of the boxes in $\mathcal{B}$. When she receives a termination proposal, $a_i^{op}(z, \mathcal{B}) \in A^{op}(z, \mathcal{B}) := \{0, 1\}$, so she can reject or accept it. The mixed stationary Markov strategy of player i is denoted by $\alpha_i = (\alpha_i^{ch}, \alpha_i^{op})$, where $\alpha_i(z, \mathcal{B}) \in \Delta A^{ch}(z, \mathcal{B}) \times \Delta A^{op}(z, \mathcal{B})$.

In state $(z, \mathcal{B})$, let $\Phi_i^{ch}(z, \mathcal{B}; \boldsymbol{a}_j)$ be the highest possible payoff that player i can achieve when she is chosen at this stage, given that player $j \neq i$ plays the Markov strategy $\boldsymbol{a}_j$. Let $\Phi_i^{op}(z, \mathcal{B}; \boldsymbol{a}_j)$ be her continuation value when she is the opponent. Also, let

$$\overline{\Phi}_i(z, \mathcal{B}; \boldsymbol{a}_j) := \frac{\Phi_i^{ch}(z, \mathcal{B}; \boldsymbol{a}_j) + \Phi_i^{op}(z, \mathcal{B}; \boldsymbol{a}_j)}{2}$$

be the average value function that accounts for the fact that each period player i is chosen with probability $1/2$. It then follows that

$$\Phi_i^{ch}(z, \mathcal{B}; \boldsymbol{a}_j) = \max_{a_i^{ch} \in \{\varnothing, T\} \cup \mathcal{B}} \begin{cases} z, & \text{if } a_i^{ch} = T \wedge \boldsymbol{a}_j^{op} = 1, \\ \delta \overline{\Phi}_i(z, \mathcal{B}; \boldsymbol{a}_j), & \text{if } a_i^{ch} = T \wedge \boldsymbol{a}_j^{op} = 0, \text{ or } a_i^{ch} = \varnothing, \\ -c_k + \delta\Big[\overline{\Phi}_i(z, \mathcal{B} \setminus b_k; \boldsymbol{a}_j) F_k(z) + \displaystyle\int_z^{+\infty} \overline{\Phi}_i(x, \mathcal{B} \setminus b_k; \boldsymbol{a}_j) dF_k(x)\Big], & \text{if } a_i^{ch} = b_k \in \mathcal{B}. \end{cases} \tag{1}$$

In words, if she proposes termination and her offer is accepted, player i receives z immediately. If her offer of termination is rejected or if she does nothing, then the next period starts, the time between periods is discounted by a factor of δ, and roles are reset. If she opens box b_k, she pays the search cost c_k immediately. Next period, contents of the box are revealed, best uncovered reward and the set of available boxes are updated, and roles are reset.

The value function of player i when she is the opponent is

$$\Phi_i^{op}(z,\mathcal{B};\boldsymbol{a}_j) = \max_{a_i^{op}\in\{0,1\}} \begin{cases} z, & \text{if } \boldsymbol{a}_j^{ch} = T \wedge a_i^{op} = 1, \\ \delta\overline{\Phi}_i(z,\mathcal{B};\boldsymbol{a}_j), & \text{if } \boldsymbol{a}_j^{ch} = T \wedge a_i^{op} = 0, \text{ or } \boldsymbol{a}_j^{ch} = \varnothing, \\ \delta\Big[\overline{\Phi}_i(z,\mathcal{B}\setminus b_k;\boldsymbol{a}_j)F_k(z) + \int_z^{+\infty} \overline{\Phi}_i(x,\mathcal{B}\setminus b_k;\boldsymbol{a}_j)dF_k(x)\Big], & \text{if } \boldsymbol{a}_j^{ch} = b_k \in \mathcal{B}. \end{cases} \tag{2}$$

When she is the opponent, player i chooses between accepting and rejecting a termination proposal. If she accepts, her payoff is z. If she rejects or if the chosen player did nothing, the next stage begins, and the state is unchanged. If the chosen player opens a box, player i observes its contents without paying the search cost.

Player i's value functions $\Phi_i^{ch}(z,\mathcal{B};\boldsymbol{\alpha}_j)$ and $\Phi_i^{op}(z,\mathcal{B};\boldsymbol{\alpha}_j)$ given player j's mixed Markov strategy $\boldsymbol{\alpha}_j$ are calculated by taking expectation of (1) and (2) with respect to $\boldsymbol{\alpha}_j(z,\mathcal{B})$.

When all boxes are open, the players collect the best uncovered reward, i.e.

$$\Phi_i^{ch}(z,\varnothing) = \Phi_i^{op}(z,\varnothing) = \overline{\Phi}_i(z,\varnothing) = z. \tag{3}$$

DEFINITION 1. *Profile of strategies $(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2)$ is a Markov perfect equilibrium if for every player $i\in\{1,2\}$ and any possible state $(z,\mathcal{B})$, if $\boldsymbol{a}_i \in \operatorname{supp}\boldsymbol{\alpha}_i$, then $\boldsymbol{a}_i^{ch}$ maximizes $\Phi_i^{ch}(z,\mathcal{B};\boldsymbol{\alpha}_j)$ and $\boldsymbol{a}_i^{op}$ maximizes $\Phi_i^{op}(z,\mathcal{B};\boldsymbol{\alpha}_j)$ subject to the boundary condition* (3).

Since the players are symmetric, I focus on the symmetric equilibria.

3. One Box

Let $\mathcal{B}$ contain just one box b and z be the safe option. Note that when initially there is only one box, z equals the initial fallback reward z_0.

Suppose that there is only one unopened box, i.e. $\mathcal{B} = \{b\}$. Let

$$S(z, F) := \mathbb{E}[\max\{z, x\}] = zF(z) + \int_{z}^{+\infty} x dF(x)$$

be the expected value of the best uncovered reward after opening the box. That is, with probability $\text{Prob}(x \leq z) = F(z)$, the reward in the box is lower than z, in which case the best uncovered option is not updated. Otherwise, the reward x discovered in the box becomes the new safe option.

Recall that, according to Weitzman (1979), social planner opens the box if and only if the expected benefit net of the search cost exceeds the safe option:

$$-c + \delta S(z, F) \geq z. \tag{SR}$$

Weitzman (1979) shows that there exists a unique z that solves the binding *social rationality* condition (SR).

Definition 2. *Reservation value $\bar{z}$ of box $b = (c, F)$ solves $-c + \delta S(\bar{z}, F) = \bar{z}$.*

Weitzman (1979) also shows that (SR) holds if and only if $z \leq \bar{z}$. Notice that if (SR) holds, then for the chosen player, doing nothing weakly dominates proposing termination.

How does the opponent respond to a termination offer? The best she can do by refusing to terminate the game is wait until she is chosen and open the box herself. Due to the discounting of future payoffs and the uncertainty regarding the period in which she is chosen, the expected payoff from opening the box is multiplied by a factor of $\frac{1}{2}\delta + \left(\frac{1}{2}\delta\right)^2 + \cdots = \frac{\delta}{2-\delta}$. Thus, the opponent rejects a termination offer if and only if the following *individual rationality* condition holds:

$$\frac{\delta}{2-\delta} \cdot [-c + \delta S(z, F)] \geq z. \tag{IR}$$

While Weitzman (1979) defines $\bar{z}$ as the threshold for opening the box, I define z^R as the threshold for rejecting a termination proposal in favor of opening the box when she is chosen.

DEFINITION 3. *Rejection threshold* z^R *of box* $b = (c, F)$ *solves* $\frac{\delta}{2-\delta} \cdot [-c + \delta S(z^R, F)] = z^R$.

It is straightforward to show that z^R is unique and that (IR) condition holds if and only if $z \leq z^R$.[2] Furthermore, $z^R \leq \bar{z}$. In other words, (IR) implies (SR), but not vice versa. Intuitively, if the box is good enough that the player is willing to wait to open it in the future, then the box is good enough that she is willing to open it today.

When $z \leq \bar{z}$, the chosen player opens the box, does nothing, or mixes between these two actions. It is easy to see that a symmetric equilibrium in *pure* stationary Markov strategies often does not exist. When a player is chosen and knows that her counterpart will open the box (do nothing) when chosen, she is better off doing nothing (opening the box). Consequently, when (SR) holds, the chosen player mixes between opening the box and doing nothing. Let π be the equilibrium probability of opening the box. The chosen player must be indifferent between *(i)* opening the box today and *(ii) someone* opening the box in the future, which translates into

$$-c + \delta S(z, F) = \frac{\pi\delta}{1 - (1-\pi)\delta} \cdot \left[-\frac{c}{2} + \delta S(z, F) \right]. \tag{4}$$

In words, her expected payoff from not opening the box today is the surplus $\delta S(z, F)$ from the box being opened eventually, less her having to pay the search cost c *half of the time* on average, infinitely discounted according to the time discount factor δ and the probability $1 - \pi$ that the box is not opened in the current period.

By solving the indifference condition above, we obtain the equilibrium probability of opening the box π as a function of the safe option z.[3] Theorem 1 summarizes the search

[2] The formal proof of this statement and all other results can be found in the appendix.

[3] For low values of c the solution may not exist because the left-hand side is always smaller. In that case, the chosen player opens the box right away. Theorem 1 accounts for that case.

and termination protocol for the model with one box.

THEOREM 1. *Let z be the safe option and $\mathcal{B} = \{b\}$. In the symmetric equilibrium,*

the chosen player

- *if $z \leq \overline{z}$, opens the box with probability*

$$\pi(z) = \begin{cases} \frac{2(1-\delta)}{\delta c} \cdot \left[-c + \delta S(z,F) \right] < 1 & \text{if } c > S(z,F) \cdot \frac{2\delta(1-\delta)}{2-\delta}, \\ 1 & \text{otherwise} , \end{cases}$$

  and does nothing with probability $1 - \pi(z)$;

- *proposes to terminate the game if $z > \overline{z}$.*

the opponent

- *rejects the termination proposal if $z \leq z^R$;*
- *accepts it otherwise.*

Comparing this to the optimal search and stopping protocol of an individual searcher, we can see that on the equilibrium path, the box is eventually opened as long as $z \leq \overline{z}$, the same cutoff as in Weitzman (1979). Put differently, the box is opened if and only if it is socially optimal to do so. If the search cost if large enough, there is a chance that the chosen player does nothing instead of opening the box. Hence, it may take several periods to open the same box that the social planner opens right away. Consequently, collaborative search results in delay as a consequence of the free riding. I measure the size of the delay and discuss comparative statics in the section that follows.

In the case of one box, we can refer to the state of the problem as just z, and the value functions take a simple form.

COROLLARY 1. *In the symmetric equilibrium, if z is the safe option and one box remains to be opened, the value functions are*

$$\Phi^{ch}(z) = \max \left\{ z, -c + \delta S(z,F) \right\},$$
$$\Phi^{op}(z) = \max \left\{ z, \frac{2-\delta}{\delta} \cdot \left[-c + \delta S(z,F) \right] \right\},$$
$$\overline{\Phi}(z) = \max \left\{ z, \frac{1}{\delta} \cdot \left[-c + \delta S(z,F) \right] \right\}.$$

Consider the case when the box is eventually opened, i.e. when $z \leq -c + \delta S(z, F)$. Notice that the chosen player can guarantee herself the payoff of an individual searcher by opening the box. Since she opens the box with a positive probability, her indifference implies that her value function is exactly that of the social planner. On average, however, each searcher is doing strictly better than the social planner since $\overline{\Phi}(z) = \frac{1}{\delta}\Phi^{ch}(z) > \Phi^{ch}(z)$. The key to understanding this result is recalling that the chosen player can wait to become an opponent, in which case her value function is strictly higher because $\Phi^{op}(z) = \frac{2-\delta}{\delta}\Phi^{ch}(z) > \Phi^{ch}(z)$. Simply speaking, each player prefers to search in a team because then there is a chance that her companion pays the search cost.

LEMMA 1. *In the symmetric equilibrium, the value functions $\Phi^{ch}(z)$ and $\overline{\Phi}(z)$ increase as*

- *the search cost c decreases,*
- *the length of the time interval between stages Δt decreases.*

The opponent's value function $\Phi^{op}(z)$ decreases in search cost; $\frac{\partial \Phi^{op}(z)}{\partial \Delta t}$ could be positive or negative.

When the search cost decreases, two effects take place. First of all, the expected net benefit of opening the box $-c + \delta S(z, F)$ increases. Secondly, the reservation value $\overline{z}$ of the box also increases, i.e. this box becomes ex-ante more attractive. As a result, the box is now opened for values of z for which it was not opened before, which drives the value functions even higher. The same arguments apply for Φ^{ch} and $\overline{\Phi}$ when the discount factor δ increases due to the shorter wait between stages of the game.

The dynamics of the opponent's value function with respect to Δt are inconclusive. On the one hand, a higher discount factor leads to a higher continuation value because the box becomes ex-ante more attractive. On the other hand, as Δt decreases, the incentive to free ride increases, and that drives the equilibrium probability of opening the box down. Consequently, the opponent's continuation value drops since it is determined by the likelihood of the chosen player opening the box. Either effect may prevail, depending on other parameters.

DELAY AND WELFARE IMPLICATIONS

Recall that it is socially optimal to open the box immediately whenever (SR) holds. According to Theorem 1, the chosen player opens the box with certainty when the search cost is low enough, and with probability less than one otherwise. Consequently, when the search cost is high enough, there is welfare loss due to the delay.

Given the time interval between stages Δt and the probability π that the box is opened each round, I define the expected delay before the box is opened as

$$D(\pi, \Delta t) = 0 \cdot \pi + \Delta t \cdot \pi \cdot (1-\pi) + 2\Delta t \cdot (1-\pi)^2 \cdot \pi + \cdots = \Delta t \cdot \frac{1-\pi}{\pi}.$$

To understand the severity of the delay in a collaborative search environment, I analyze how the equilibrium probability $\pi(z)$ of opening the box varies with the search cost c and the safe option z.

Recall that by Theorem 1, $\pi(z)$ equals to one (meaning that there is no delay) for low enough values of the search cost, and is between zero and one (there is delay) when the search costs are sufficiently high. In particular,[4]

- if $c > \bar{c} := S(\bar{z}, F) \cdot \frac{2\delta(1-\delta)}{(2-\delta)}$, the search costs are so large that there is an interior solution $\pi \in (0,1)$ for every $z \in [0, \bar{z}]$;
- if $c < \underline{c} := S(0, F) \cdot \frac{2\delta(1-\delta)}{(2-\delta)}$, the search costs are so small that opening the box right away is strictly dominant for all $z \in [0, \bar{z}]$;
- if $c \in [\underline{c}, \bar{c}]$, then there is an interior solution $\pi(z) \in (0,1)$ for $z < \tilde{z}$ that solves $c = S(\tilde{z}, F) \cdot \frac{2\delta(1-\delta)}{(2-\delta)}$ and the box is opened right away for $z \geq \tilde{z}$.

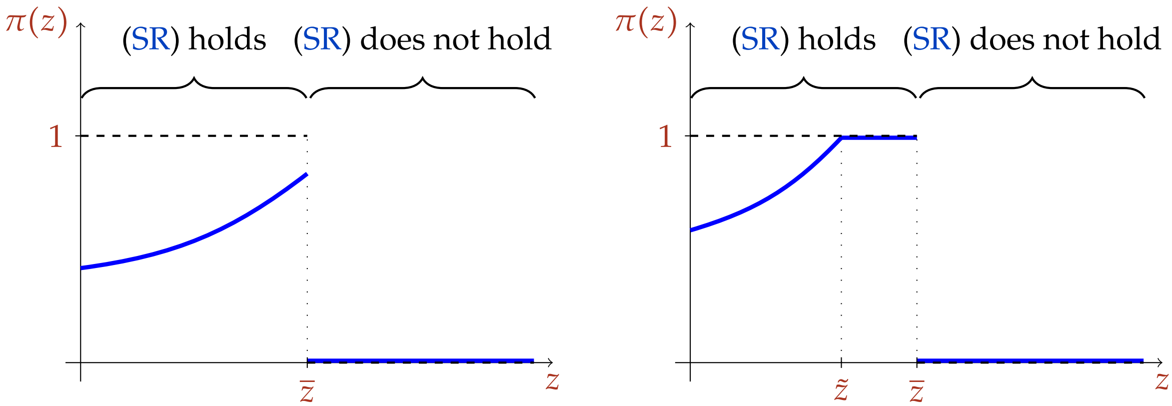
The properties of $\pi(z)$ are summarized in Lemma 2 and illustrated in Figure 1.

LEMMA 2. *Let $z \leq \bar{z}$ be the safe option and $\mathcal{B} = \{b\}$. In the symmetric equilibrium, the probability that the chosen player opens the box $\pi(z)$ has the following properties.*

1. *If $c \geq \bar{c}$, then $\pi(z) \in (0,1)$ and is strictly increasing and strictly convex;*
2. *if $c \leq \underline{c}$, then $\pi(z) = 1$;*
3. *if $c \in (\underline{c}, \bar{c}_k)$, then $\pi(z)$ exhibits the same properties as in case (1) for $z \in [0, \tilde{z}]$, and as in case (2) for $z \in [\tilde{z}, \bar{z}]$.*

Strikingly, the equilibrium probability opening the box is *increasing* in the value of the

[4] Note that $S(z, F)$ is strictly increasing and convex, since $S(z, F) = \mathbb{E}[\max\{z, x\}] = zF(z) + \int_z^{+\infty} x dF(x)$, it follows that $\frac{\partial S(z,F)}{\partial z} = F(z) > 0$ and $\frac{\partial^2 S(z,F)}{\partial z^2} = f(z) > 0$.



(a) $c \geq \bar{c}$. *(b)* $c \in (\underline{c}, \bar{c})$.

Figure 1. *The probability $\pi(z)$ that the chosen player opens the box in the symmetric equilibrium. The dashed black lines represent the efficient level of $\pi(z)$.*

safe option at an increasing rate. The key to understanding this result is recalling that the safe option z is not the outside option for the chosen player because she cannot unilaterally deviate to collect it (her opponent must accept the termination offer). The effective outside option is to open the box today, and increasing z makes it more appealing to do so. Once the box is opened, the game ends, and *at least* z is collected. To remain indifferent between opening the box and not, the chosen player must rationally expect that the box is more likely to be opened in the future, which drives $\pi(z)$ up and the delay down.

Notice the lowest delay occurs at $z = \bar{z}$, when the chosen player is indifferent between *(i)* opening the box, *(ii)* not opening the box, and *(iii)* collecting the safe option $\bar{z}$ (if she could). At that point, the expected benefit from opening the box $-c + \delta S(z, F)$ is the highest among all $z \in [0, \bar{z}]$ which, by the logic described above, makes opening the box sooner more desirable.[5] Delay is the highest when the safe option is zero when there seems to be the most to gain from opening the box. However, the expected reward from opening the box is actually the lowest compared to higher safe options. Thus, the indifference condition dictates that the box is the least likely to be opened.

Next, I discuss the comparative statics of the expected delay with respect to various model parameters. In general, as the value of opening the box today increases, so does the continuation value of not opening the box today. To remain indifferent, the chosen player must rationally expect a higher probability of the box being opened in the future, as

[5]For every $z < \bar{z}$ it is true that $-c + \delta S(z, F) < -c + \delta S(\bar{z}, F) = \bar{z}$.

prescribed by equation (4). Hence, *less delay* is associated with *higher reservation values* due to

- lower search cost c: the root of free riding lies in the unwillingness to pay the search cost. Reducing the search cost reduces the incentive to do nothing when chosen;
- "better" rewards: changing $F(\cdot)$ to $G(\cdot)$ such that $G(x) \leq F(x)\ \forall x$ leads to a higher expected reward $S(z, G)$ and higher π. Performing a mean-preserving spread on $F(\cdot)$ (making the box riskier) has the same effect.

The dynamic of the equilibrium probability of opening the box π with respect to the time between stages Δt is inconclusive. Recall that decreasing Δt increases the discount factor δ. On the one hand, increasing δ increases the value of opening the box today, and by the argument discussed above, decreases the delay. On the other hand, higher δ increases the players' willingness to wait for their opponent to perform the search, which drives the delay up. Either effect may prevail.

4. Many Boxes

Let $\boldsymbol{b}$ be the box with the highest reservation value $\overline{\boldsymbol{z}}$ among the unopened boxes in $\mathcal{B}$, i.e.

$$\boldsymbol{b} := \arg\max_{b_k \in \mathcal{B}} \overline{z}_k \text{ and } \overline{\boldsymbol{z}} := \max_{b_k \in \mathcal{B}} \overline{z}_k.$$

In state $(z, \mathcal{B})$, if $z > \overline{\boldsymbol{z}}$, then opening *any* leads to the highest payoff for the chosen player. As such, any termination proposal in this state is accepted. Next, suppose $z \leq \overline{\boldsymbol{z}}$. When does player i reject a termination proposal?

To reject the termination proposal, player i must expect that the value of continuing the game and opening some boxes exceeds z. When player i rejects the proposal, she is chosen next period with probability $1/2$, and with probability $1/2$ she faces another termination proposal.[6] Because player i faces a termination proposal whenever she is not chosen, her problem is effectively the problem of an individual searcher who discounts her payoff with

[6]When termination offer is rejected, next stage begins with the same state of the problem. Since we are considering stationary Markov strategies, if player j proposes termination in this period, she also proposed termination in the next period.

a factor of $\delta/(2-\delta)$, instead of δ. According to Weitzman (1979), player i's optimal policy is to open the boxes in the order of decreasing reservation value. When the highest reservation value becomes less than the maximum observed reward, she proposes termination, and her opponent agrees. Theorem 2 summarizes the best response of the opponent to a termination proposal.

THEOREM 2. *Player i's best response to player j's termination proposal in state $(z, \mathcal{B})$ is to*

- *reject it if and only if $z \leq \delta\overline{\Phi}_i(z, \mathcal{B})$,*
- *accept it otherwise,*

where for any state $(\tilde{z}, \tilde{\mathcal{B}})$ such that $\tilde{z} \geq z$ and $\tilde{\mathcal{B}} \subseteq \mathcal{B}$, player i's discounted average value function $\overline{\Phi}_i$ is recursively defined by

$$\delta\overline{\Phi}_i(\tilde{z}, \tilde{\mathcal{B}}) = \frac{\delta}{2-\delta} \max\left\{\tilde{z}, \max_{b_k \in \tilde{\mathcal{B}}}\left(-c_k + \delta\left[\overline{\Phi}_i(\tilde{z}, \tilde{\mathcal{B}} \smallsetminus b_k)F_k(\tilde{z}) + \int_z^{+\infty} \overline{\Phi}_i(x, \tilde{\mathcal{B}} \smallsetminus b_k)dF_k(x)\right]\right)\right\},$$

$$\overline{\Phi}_i(\tilde{z}, \varnothing) = \tilde{z}.$$

Notice that there is no explicit solution for when to accept the termination proposal, unlike in the one-box case. To make her decision, player i needs to iterate the search process forward until no more boxes are left. That said, when calculating the expected value of rejecting the proposal, she uses the socially optimal search protocol. The reason is that her discounted average value function $\delta\overline{\Phi}_i$ satisfies the Bellman equation of an individual searcher with discount factor $\delta/(2-\delta)$. Thus, when player j proposes termination whenever she is chosen, player i opens boxes in the order of decreasing reservation values, and proposes termination when the social planner would. That offer is accepted because it maximizes player j's payoff.

Two simple arguments provide sufficient conditions for when player i accepts and rejects the termination proposal.

LEMMA 3. *In state $(z, \mathcal{B})$, let $\boldsymbol{b}$ be the box with the highest reservation value $\overline{z}$ among the unopened boxes in $\mathcal{B}$ and let $\boldsymbol{z}^R$ be the rejection cutoff of that box. Then,*

- *$z \leq \boldsymbol{z}^R$ is sufficient to reject the termination offer,*
- *$z > \overline{z}$ is sufficient to accept the termination offer.*

Recall that $\boldsymbol{b}$ is the "best" unopened box. Intuitively, if z is low enough that player i wants to wait and open just box $\boldsymbol{b}$, then she rejects the termination proposal. Conversely, if z is high enough that player i does not want to open every box $\boldsymbol{b}$ when she is eventually chosen, then she rejects the termination proposal.

Next, I consider the problem of the chosen player. If $z > \overline{\boldsymbol{z}}$, proposing termination is the weakly dominant strategy. In this case, her termination proposal is accepted by Lemma 3, the game ends, and both players receive z.

On the other hand, if $z \leq \overline{\boldsymbol{z}}$, then proposing termination is weakly dominated by doing nothing. Similarly to the one-box case, let us look for a symmetric mixed-strategy equilibrium. Given player j's mixed strategy, player i should be indifferent between every action she plays with positive probability. In particular, to be indifferent between opening two or more boxes, player i should expect the same average continuation value after opening each of them. However, according to Weitzman (1979), the value of opening the box with the highest continuation value exceeds the value of opening any other box.

Consequently, player i can only be indifferent between opening box $\boldsymbol{b}$ and doing nothing. When player i opens box $\boldsymbol{b}$, she pays the search cost $\boldsymbol{c}$, and moves on to the next stage of the problem with reset roles, fewer boxes, and a potentially higher uncovered reward. When she does nothing, next stage begins, roles are reset, but the state of the problem remains the same. As a result, the probability of opening box $\boldsymbol{b}$ solves player i's indifference condition between these two options, given that player j plays the same mixed strategy. The chosen player's search protocol in the mixed-strategy symmetric equilibrium is described in Theorem 3.

THEOREM 3. *In state $(z, \mathcal{B})$, let $\boldsymbol{b} = (\boldsymbol{c}, \boldsymbol{F})$ be the box with the highest reservation value $\overline{\boldsymbol{z}}$ among the unopened boxes in $\mathcal{B}$. In the symmetric equilibrium, the chosen player*

- *if $z \leq \overline{\boldsymbol{z}}$,*
  - *opens box $\boldsymbol{b}$ with probability $\boldsymbol{\pi}(z, \mathcal{B}) = \min\left\{\frac{2(1-\delta)}{\delta \boldsymbol{c}} \cdot \left[-\boldsymbol{c} + \delta \overline{\Phi}^{\boldsymbol{b}}(z, \mathcal{B})\right], 1\right\}$,*
  - *does nothing otherwise;*
- *proposes termination if $z > \overline{\boldsymbol{z}}$.*

Here, in state $(\tilde{z}, \tilde{\mathcal{B}})$ such that $\tilde{z} \geq z$ and $\tilde{\mathcal{B}} \subseteq \mathcal{B}$, the average value $\overline{\Phi}$ satisfies

$$\delta\overline{\Phi}(\tilde{z}, \tilde{\mathcal{B}}) = \max\Big\{\tilde{z}, \max_{b_k \in \tilde{\mathcal{B}}}\big(-c_k + \delta\overline{\Phi}^{b_k}(\tilde{z}, \tilde{\mathcal{B}})\big)\Big\},$$
$$\overline{\Phi}(\tilde{z}, \varnothing) = \tilde{z},$$

and the average value $\overline{\Phi}^{b_k}$ after opening box $b_k = (c_k, F_k) \in \tilde{\mathcal{B}}$ is

$$\overline{\Phi}^{b_k}(\tilde{z}, \tilde{\mathcal{B}}) := \overline{\Phi}(\tilde{z}, \tilde{\mathcal{B}} \smallsetminus b_k)F_k(\tilde{z}) + \int_{z}^{+\infty} \overline{\Phi}_i(x, \tilde{\mathcal{B}} \smallsetminus b_k)dF_k(x).$$

Recall that in Weitzman (1979), the individual searcher makes decisions myopically: at each stage of the search process, she compares the reservation value of the best unopened box to the highest reward uncovered so far. With two searchers, this is no longer true. Both the chosen player and the opponent take the future into account when making decisions: the chosen player needs the future value function to calculate $\boldsymbol{\pi}$, while the opponent needs it to calculate the acceptance cutoff rule. At the same time, the *order* and the *stopping rule* on the equilibrium path are identical to that of Weitzman (1979): the box with the highest reservation value is opened next, and the game is terminated when the best uncovered reward exceeds the reservation value of all unopened boxes. The only difference is that, whenever $\boldsymbol{\pi} < 1$, box $\boldsymbol{b}$ is opened with a delay. Delay occurs because each player hopes that her opponent ends up opening the box and paying the search cost. Depending on the size of the search costs, *every* box may be opened with a delay.

While Theorem 3 implicitly defines $\boldsymbol{\pi}(z, \mathcal{B})$, the equilibrium probability that the chosen player opens box $\boldsymbol{b}$, the one-box case provides a convenient lower bound. In particular, it is straightforward to show that $\overline{\Phi}^{\boldsymbol{b}}(z, \mathcal{B}) \geq \overline{\Phi}^{\boldsymbol{b}}(z, \{\boldsymbol{b}\}) = S(z, F)$, meaning that the average value after opening box $\boldsymbol{b}$ is higher if $\boldsymbol{b}$ is not the only box left to open. Consequently, $\boldsymbol{\pi}(z, \mathcal{B}) \geq \boldsymbol{\pi}(z, \{\boldsymbol{b}\})$, where $\boldsymbol{\pi}(z, \{\boldsymbol{b}\})$ is the equilibrium probability of opening box $\boldsymbol{b}$ when $\boldsymbol{b}$ is the only box to be opened. Theorem 1 describes the one-box case and provides an explicit expression for $\boldsymbol{\pi}(z, \{\boldsymbol{b}\}) = \pi(z)$. This lower bound is useful for determining whether the box is opened with a delay or not. In particular, if $\boldsymbol{c} \leq S(z, \boldsymbol{F}) \cdot \frac{2\delta(1-\delta)}{2-\delta}$, Theorem 1 concludes that box $\boldsymbol{b}$ is opened without a delay. In that case, the chosen player prefers to open the box immediately, rather than wait for her counterpart to exert the low enough search cost.

Next, let us examine players' value functions.

COROLLARY 2. *In state $(z, \mathcal{B})$, let $\boldsymbol{b} = (\boldsymbol{c}, \boldsymbol{F})$ be the box with the highest reservation value $\overline{\boldsymbol{z}}$ among the unopened boxes in $\mathcal{B}$. In the symmetric equilibrium, the value functions are*

$$\begin{aligned}\Phi^{ch}(z,\mathcal{B}) &= \max\left\{z, -\boldsymbol{c} + \delta\overline{\Phi}^{\boldsymbol{b}}(z,\mathcal{B})\right\},\\ \Phi^{op}(z,\mathcal{B}) &= \max\left\{z, \frac{2-\delta}{\delta}\cdot\left[-\boldsymbol{c} + \delta\overline{\Phi}^{\boldsymbol{b}}(z,\mathcal{B})\right]\right\},\\ \overline{\Phi}(z,\mathcal{B}) &= \max\left\{z, \frac{1}{\delta}\cdot\left[-\boldsymbol{c} + \delta\overline{\Phi}^{\boldsymbol{b}}(z,\mathcal{B})\right]\right\},\end{aligned}$$

where for any state $(\tilde{z}, \tilde{\mathcal{B}})$ such that $\tilde{z} \geq z$ and $\tilde{\mathcal{B}} \subseteq \mathcal{B}$, the average value $\overline{\Phi}^{b_k}$ after opening box $b_k = (c_k, F_k) \in \tilde{\mathcal{B}}$ is

$$\overline{\Phi}^{b_k}(\tilde{z},\tilde{\mathcal{B}}) = \overline{\Phi}(\tilde{z},\tilde{\mathcal{B}}\setminus b_k)F_k(\tilde{z}) + \int_{z}^{+\infty}\overline{\Phi}_i(x,\tilde{\mathcal{B}}\setminus b_k)dF_k(x).$$

When chosen, each player does as well as an individual searcher because she bears the search cost whenever she decides to open a box. When not chosen, each player does better than an individual searcher because if a box is opened, she does not pay the search cost. As a result, on average, each player prefers to search with a colleague rather than on her own.

Recall from Section 3 that boxes with higher reservation values are opened faster. In the multi-box setting, boxes with the higher reservation value are opened earlier rather than later, suggesting that the delay grows over the search process. On the other hand, the delay also decreases with the value of the outside option z, which grows as more rewards are uncovered. In the end, it is unclear whether the players shirk more or less as the search process goes on.

5. CONCLUSION

This paper examined a model of sequential search for a public good performed by a team of agents. I showed that group search results in the socially optimal search and stopping rule. However, delay occurs at every stage of the learning process because agents free ride. Overall, the team manager prefers to delegate research to individual agents, but each agent prefers to search with a teammate.

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Appendix: Omitted Proofs

Lemma 4. *Let* $H(z,b) := \frac{\delta}{2-\delta} \cdot [-c + S(z,F)]$ *and let* z^R *solve* $H(z^R,b) = z^R$*. Then,* (IR) *holds if and only if* $z \leq z^R$*.*

Proof. Since $\frac{\partial S(z,F)}{\partial z} = F(z) \in [0,1]$, $H(z,b)$ is increasing in z at the rate less than one. Since $z^R = H(z^R,b)$, $z \leq H(z,b)$ if and only if $z \leq z^R$. □

Proof of Theorem 1 and Corollary 1

With one box, for each z we have two states of the world, the box being closed and the box being open. With a slight abuse of notation, below I call z the state of the world when the box is closed. Once the box is open, boundary condition (3) states that both value functions equal z.

When player i faces a termination proposal, that is, when player j plays $\boldsymbol{a}_j$ such that $\boldsymbol{a}_j^{ch} = T$, her Bellman equation in state z is

$$\Phi_i^{op}(z;\boldsymbol{a}_j) = \max_{a_i^{op} \in \{0,1\}} \begin{cases} z, & \text{if } a_i^{op} = 1, \\ \delta \overline{\Phi}_i(z;\boldsymbol{a}_j), & \text{if } a_i^{op} = 0. \end{cases}$$

To calculate her average value function $\overline{\Phi}_i$, we first calculate her value function when she is chosen as

$$\Phi_i^{ch}(z;\boldsymbol{a}_j) = \max_{a_i^{ch}\in\{\varnothing,T,b\}} \begin{cases} z, & \text{if } a_i^{ch} = T \wedge \boldsymbol{a}_j^{op} = 1, \\ \delta\overline{\Phi}_i(z;\boldsymbol{a}_j), & \text{if } a_i^{ch} = T \wedge \boldsymbol{a}_j^{op} = 0, \text{ or } a_i^{ch} = \varnothing, \\ -c + \delta S(z,F), & \text{if } a_i^{ch} = b. \end{cases}$$

Player i rejects a termination proposal if and only if $\delta\overline{\Phi}_i(z;\boldsymbol{a}_j) \geq z$, that is, if she expects a higher payoff from continuing the game than from terminating it. Since player j is not helping her open the box (she plays $\boldsymbol{a}_j^{ch} = T$), player i would only reject the termination proposal if she expects to open the box herself when chosen, i.e. $\Phi_i^{ch}(z;\boldsymbol{a}_j) = -c + \delta S$. Then, using the facts that *(i)* it may take her time to get chosen, $\Phi_i^{op}(z;\boldsymbol{a}_j) = \delta\overline{\Phi}_i(z;\boldsymbol{a}_j)$, and *(ii)* her average value equals $\overline{\Phi}_i = \frac{\Phi^{ch}+\Phi^{op}}{2}$, we get that player i rejects the termination offer if and only if

$$\frac{\delta}{2-\delta}\cdot[-c+\delta S(z,F)] \geq z.$$

Note that the inequality above is the (IR) condition and it holds if and only if $z \leq z^R$ according to Lemma 4.

Next consider the decision of player i when she is chosen. First suppose that the social rational condition (SR) does not hold, i.e. $z > \overline{z}$. In this case, player i does not want to open the box, she would rather propose termination, since she knows that the offer will be accepted. As a result, $\Phi_i^{ch}(z) = \Phi_i^{op}(z) = z$.

Next suppose that (SR) holds, i.e. $z \leq -c + \delta S(z,F)$. Player i's value function when she is chosen becomes

$$\Phi_i^{ch}(z;\boldsymbol{a}_j) = \max_{a_i^{ch}\in\{\varnothing,b\}} \begin{cases} \delta\overline{\Phi}_i(z;\boldsymbol{a}_j), & \text{if } a_i^{ch} = \varnothing, \\ -c + \delta S(z,F), & \text{if } a_i^{ch} = b, \end{cases}$$

since her payoff from proposing termination is weakly worse than her payoff of opening the box. When she is the opponent, her value function is

$$\Phi_i^{op}(z;\boldsymbol{a}_j) = \begin{cases} \delta\overline{\Phi}_i(z;\boldsymbol{a}_j), & \text{if } \boldsymbol{a}_j^{ch} = \varnothing, \\ \delta S(z,F), & \text{if } \boldsymbol{a}_j^{ch} = b. \end{cases}$$

Let π_j be player j's mixed strategy when she is chosen. $\pi_j(z)$ denotes the probability that player j opens the box (i.e. plays action $\boldsymbol{a}_j^{ch} = b$) in state z. Then, player i's value function when she is the opponent is

$$\Phi_i^{op}(z;\pi_j) = \pi_j \cdot \delta S(z,F) + (1-\pi_j) \cdot \delta\overline{\Phi}_i(z;\pi_j).$$

For player i to play a mixed strategy in the symmetric equilibrium, she must be indifferent between opening the box and doing nothing, i.e.

$$\Phi_i^{ch}(z;\pi_j) = \delta\overline{\Phi}_i(z;\pi_j) = -c + \delta S(z,F).$$

Solving these equations given that $\overline{\Phi}_i = \frac{\Phi^{ch}+\Phi^{op}}{2}$, we get that in the symmetric equilibrium

$$\Phi^{ch}(z) = \delta\overline{\Phi}(z) = -c + \delta S(z,F), \quad \Phi^{op}(z) = \frac{2-\delta}{\delta} \cdot [-c + \delta S(z,F)],$$
$$\pi(z) = \frac{2(1-\delta)}{\delta c} \cdot [-c + \delta S(z,F)].$$

Note that there is no interior solution $\pi \in (0,1)$ when opening the box strictly dominates doing nothing, which happens if and only if

$$-c + \delta S(z,F) > \delta\overline{\Phi}_i(z;\pi_j) \text{ and } \Phi_i^{op}(z;\pi_j) = \delta S(z,F) \iff c < S(z,F) \cdot \frac{2\delta(1-\delta)}{2-\delta}.$$

Summarizing our findings for the cases when (SR) does and does not hold, the value functions in the symmetric equilibrium are

$$\Phi^{ch}(z) = \max\{z, -c + \delta S(z,F)\},$$
$$\Phi^{op}(z) = \max\left\{z, \frac{2-\delta}{\delta} \cdot [-c + \delta S(z,F)]\right\},$$
$$\overline{\Phi}(z) = \max\left\{z, \frac{1}{\delta} \cdot [-c + \delta S(z,F)]\right\}.$$

Proof of Lemma 1

Firstly, let us find how the reservation value $\bar{z}$ changes with c and δ (since Δt enters expressions via $\delta = e^{-r\Delta t}$ only). Recall that $\bar{z}$ is implicitly defined by

$$\bar{z} = -c + \delta S(z, F).$$

Differentiating this equation with respect to the variables of interest, we get

$$\frac{\partial \bar{z}}{\partial c} = -1 + \delta F(\bar{z}) \cdot \frac{\partial \bar{z}}{\partial c} \Rightarrow \frac{\partial \bar{z}}{\partial c} = \frac{-1}{1 - \delta F(\bar{z})} < 0,$$

$$\frac{\partial \bar{z}}{\partial \delta} = S(z, F) + \delta F(\bar{z}) \cdot \frac{\partial \bar{z}}{\partial \delta} \Rightarrow \frac{\partial \bar{z}}{\partial \delta} = \frac{S(z, F)}{1 - \delta F(\bar{z})} > 0,$$

using the fact that $\frac{\partial S(z)}{\partial z} = F(z)$.

Since the reservation value increases as c decreases and δ increases, the condition for opening the box $z \le \bar{z}$ is met for a wider range of z. Since $\frac{\partial \Phi^{ch}(z)}{\partial \delta} = S(z, F) > 0$ and $\frac{\partial \overline{\Phi}(z)}{\partial \delta} = \frac{c}{\delta^2} > 0$ when $z \le \bar{z}$, it follows directly that $\Phi^{ch}(z)$ and $\overline{\Phi}(z)$ increase. The same argument can be applied for dynamics of $\Phi^{op}(z)$ with respect to c. The sign of $\frac{\partial \Phi^{op}(z)}{\partial \delta}$ is inconclusive.

Proof of Lemma 2

When there is an interior solution,

$$\pi(z) = \frac{2(1-\delta)}{\delta c} \cdot [-c + \delta S(z, F)].$$

Then,

$$\frac{\partial \pi(z)}{\partial z} = \frac{2(1-\delta)}{c} \cdot F(z) > 0, \text{ and } \frac{\partial^2 \pi(z)}{\partial z^2} = \frac{2(1-\delta)}{c} \cdot f(z) > 0.$$

Proof of Theorem 2

When player i faces a termination proposal, that is, when player j plays $\boldsymbol{a}_j$ such that $\boldsymbol{a}_j^{ch} = T$ and $\boldsymbol{a}_j^{op} = 1$ whenever $z > \overline{z}$, her Bellman equation in state $(z, \mathcal{B})$ is

$$\Phi_i^{op}(z,\mathcal{B};\boldsymbol{a}_j) = \max_{a_i^{op}\in\{0,1\}} \begin{cases} z, & \text{if } a_i^{op} = 1, \\ \delta\overline{\Phi}_i(z,\mathcal{B};\boldsymbol{a}_j), & \text{if } a_i^{op} = 0. \end{cases}$$

Consequently, player i rejects the termination proposal if and only if $z \leq \delta\overline{\Phi}_i(z,\mathcal{B};\boldsymbol{a}_j)$. To calculate her average value function $\overline{\Phi}_i$, we first calculate her value function when she is chosen:

$$\Phi_i^{ch}(z,\mathcal{B};\boldsymbol{a}_j) = $$
$$\max_{a_i^{ch}\in\{\varnothing,T\}\cup\mathcal{B}} \begin{cases} z, & \text{if } a_i^{ch} = T \wedge z > \overline{z}, \\ \delta\overline{\Phi}_i(z,\mathcal{B};\boldsymbol{a}_j), & \text{if } a_i^{ch} = T \wedge z \leq \overline{z}, \text{ or } a_i^{ch} = \varnothing, \\ -c_k + \delta\left[\overline{\Phi}_i(z,\mathcal{B}\setminus b_k;\boldsymbol{a}_j)F_k(z) + \int_z^{+\infty} \overline{\Phi}_i(x,\mathcal{B}\setminus b_k;\boldsymbol{a}_j)dF_k(x)\right], & \text{if } a_i^{ch} = b_k. \end{cases}$$

Using the definition of player i's average value function, $\overline{\Phi}_i = \frac{\Phi^{ch}+\Phi^{op}}{2}$, we get that

$$\delta\overline{\Phi}_i(z,\mathcal{B};\boldsymbol{a}_j) = \frac{\delta}{2-\delta}\max\begin{cases} z, \\ \max_{b_k\in\mathcal{B}}\left(-c_k + \delta\left[\overline{\Phi}_i(z,\mathcal{B}\setminus b_k;\boldsymbol{a}_j)F_k(z) + \int_z^{+\infty}\overline{\Phi}_i(x,\mathcal{B}\setminus b_k;\boldsymbol{a}_j)dF_k(x)\right]\right). \end{cases}$$

The equation above with boundary condition (3) allow us to calculate $\delta\overline{\Phi}_i(z,\mathcal{B};\boldsymbol{a}_j)$ implicitly. Player i's best response to a termination proposal in state $(z,\mathcal{B})$ is to reject it if and only if $z \leq \delta\overline{\Phi}_i(z,\mathcal{B};\boldsymbol{a}_j)$.

Proof of Lemma 3

Letting $\boldsymbol{b} = (\boldsymbol{c},\boldsymbol{F})$, we have

- $z > \overline{z} \iff z > -\boldsymbol{c} + \delta S(z,\boldsymbol{F})$, meaning that $\delta\overline{\Phi}_i(z,\mathcal{B}) = \frac{\delta}{2-\delta}\cdot z$ and $\Phi_i^{op} = z$, and the termination offer is accepted;

- $z \leq \boldsymbol{z}^R \iff z \leq \frac{\delta}{2-\delta} \cdot [-\boldsymbol{c} + \delta S(z, \boldsymbol{F})]$, meaning that

$$\frac{\delta}{2-\delta} \cdot \Big\{ -\boldsymbol{c} + \delta\Big[\overline{\Phi}_i(z, \mathcal{B} \smallsetminus \boldsymbol{b})\boldsymbol{F}(z) + \int_z^{+\infty} \overline{\Phi}_i(x, \mathcal{B} \smallsetminus \boldsymbol{b}) d\boldsymbol{F}(x)\Big]\Big\} \geq \frac{\delta}{2-\delta} \cdot [-\boldsymbol{c} + \delta S(z, \boldsymbol{F})] \geq z,$$

and the termination offer is rejected in favor of opening at least one box.

Proof of Theorem 3 and Corollary 2

Consider the problem of the chosen player. If $z > \overline{z}$, proposing termination is the weakly dominant action. In this case, her termination proposal is accepted by Lemma 3, game ends, and both players receive z.

Next suppose that $z \leq \overline{z}$. Player i's value function when she is chosen is

$$\Phi_i^{ch}(z, \mathcal{B}; \boldsymbol{a}_j) = \max_{a_i^{ch} \in \varnothing \cup \mathcal{B}} \begin{cases} \delta\overline{\Phi}_i(z, \mathcal{B}; \boldsymbol{a}_j), & \text{if } a_i^{ch} = \varnothing, \\ -c_k + \delta\overline{\Phi}_i^{b_k}(z, \mathcal{B}; \boldsymbol{a}_j), & \text{if } a_i^{ch} = b_k \in \mathcal{B}, \end{cases}$$

where

$$\overline{\Phi}_i^{b_k}(z, \mathcal{B}; \boldsymbol{a}_j) := \overline{\Phi}_i(z, \mathcal{B} \smallsetminus b_k; \boldsymbol{a}_j) F_k(z) + \int_z^{+\infty} \overline{\Phi}_i(x, \mathcal{B} \smallsetminus b_k; \boldsymbol{a}_j) dF_k(x)$$

denotes the average value function after opening box $b_k \in \mathcal{B}$.

When player i is the opponent, her value function is

$$\Phi_i^{op}(z, \mathcal{B}; \boldsymbol{a}_j) = \begin{cases} \delta\overline{\Phi}_i(z, \mathcal{B}; \boldsymbol{a}_j), & \text{if } \boldsymbol{a}_j^{ch} = \varnothing, \\ \delta\overline{\Phi}_i^{b_k}(z, \mathcal{B}; \boldsymbol{a}_j), & \text{if } \boldsymbol{a}_j^{ch} = b_k \in \mathcal{B}. \end{cases}$$

Let π_j be player j's mixed strategy when she is chosen. $\pi_j^k(z, \mathcal{B})$ denotes the probability that player j opens box $b_k \in \mathcal{B}$ and $\pi_j^{\varnothing}(z, \mathcal{B})$ denotes the probability that she does nothing. Then, player i's value function when she is the opponent is

$$\Phi_i^{op}(z, \mathcal{B}; \pi_j) = \pi_j^{\varnothing}(z, \mathcal{B}) \cdot \delta\overline{\Phi}_i(z, \mathcal{B}; \boldsymbol{a}_j) + \sum_{b_k \in \mathcal{B}} \pi_j^k(z, \mathcal{B}) \cdot \delta\overline{\Phi}_i^{b_k}(z, \mathcal{B}; \boldsymbol{a}_j).$$

To play a mixed strategy π_i, player i must be indifferent between all the actions that she plays with positive probability. i.e.

$$\delta\overline{\Phi}_i(z,\mathcal{B};\boldsymbol{a}_j) = -c_k + \delta\overline{\Phi}_i^{b_k}(z,\mathcal{B};\boldsymbol{a}_j), \text{ for all } b_k \in \mathcal{B} \text{ such that } \pi_i^k(z,\mathcal{B}) > 0.$$

Recall that according to Weitzman (1979), when facing two boxes with different reservation values, strictly prefers to open the box with the higher reservation value first. Consequently, in the symmetric equilibrium, player i cannot be indifferent between opening all the boxes. She can only be indifferent between doing nothing and opening the box with the highest reservation value. Let $\boldsymbol{\pi}_j > 0$ be the probability that player j opens box $\boldsymbol{b}$ and $1 - \boldsymbol{\pi}_i$ be the probability that she does nothing. Player i's value functions become

$$\Phi_i^{op}(z,\mathcal{B};\boldsymbol{\pi}_j) = (1-\boldsymbol{\pi}_j)\cdot\delta\overline{\Phi}_i(z,\mathcal{B};\boldsymbol{\pi}_j) + \boldsymbol{\pi}_j\cdot\delta\overline{\Phi}_i^{\boldsymbol{b}}(z,\mathcal{B};\boldsymbol{\pi}_j),$$

$$\Phi_i^{ch}(z,\mathcal{B};\boldsymbol{\pi}_j) = \delta\overline{\Phi}_i(z,\mathcal{B};\boldsymbol{\pi}_j) = -\boldsymbol{c} + \delta\overline{\Phi}_i^{\boldsymbol{b}}(z,\mathcal{B};\boldsymbol{\pi}_j).$$

Solving these equations, we get that in the symmetric equilibrium each player opens box $\boldsymbol{b}$ with probability

$$\boldsymbol{\pi}(z,\mathcal{B}) = \min\left\{\frac{2(1-\delta)}{\delta\boldsymbol{c}}\cdot\left[-\boldsymbol{c}+\delta\overline{\Phi}^{\boldsymbol{b}}(z,\mathcal{B})\right], 1\right\}.$$

Summarizing our findings for the cases when (SR) does and does not hold for box $\boldsymbol{b}$, the value functions in the symmetric equilibrium are

$$\begin{aligned}
\Phi^{ch}(z,\mathcal{B}) &= \max\left\{z, -\boldsymbol{c}+\delta\overline{\Phi}^{\boldsymbol{b}}(z,\mathcal{B})\right\},\\
\Phi^{op}(z,\mathcal{B}) &= \max\left\{z, \frac{2-\delta}{\delta}\cdot\left[-\boldsymbol{c}+\delta\overline{\Phi}^{\boldsymbol{b}}(z,\mathcal{B})\right]\right\},\\
\overline{\Phi}(z,\mathcal{B}) &= \max\left\{z, \frac{1}{\delta}\cdot\left[-\boldsymbol{c}+\delta\overline{\Phi}^{\boldsymbol{b}}(z,\mathcal{B})\right]\right\}.
\end{aligned}$$