Comparative Statics and Envelope Theorems

Lecture 8

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Econ 8000 — Math Camp

Motivation

Lectures 6 and 7 solve one optimization problem at a time.

Economics rarely stops there: the interesting question is usually how the solution and its value respond when a parameter moves.

- What does a tax do to the equilibrium price?
- What does an extra unit of capacity add to profit?

The solved problem almost never has a closed form, so the answers must be read from the equations that characterize the solution.

This method is called *comparative statics*, and its main tool is the *implicit function theorem*.

The implicit function theorem

Differentiating through an equation

Let $F : U \to \mathbb{R}$ be C^1 on an open set $U \subseteq \mathbb{R}^2$, with the arguments written $F(x;\theta)$.

- An equation in the endogenous variable x, indexed by the parameter θ.

A *solution branch* is a differentiable $x(\theta)$ that solves the equation identically,

$$F\big(x(\theta);\theta\big) = 0.$$

Differentiating that identity by the chain rule of Lecture 4,

$$F_x\big(x(\theta);\theta\big)\, x'(\theta) + F_\theta\big(x(\theta);\theta\big) = 0, \qquad \text{so} \qquad x'(\theta) = -\frac{F_\theta\big(x(\theta);\theta\big)}{F_x\big(x(\theta);\theta\big)} \quad \text{where } F_x \neq 0.$$

Solving for $x'(\theta)$ requires $F_x\big(x(\theta);\theta\big) \neq 0$.

From one equation to a system

The same logic runs for systems.

Let $X \subseteq \mathbb{R}^n$ and $\Theta \subseteq \mathbb{R}^r$ be open.

Let $F : X \times \Theta \to \mathbb{R}^n$ have

- n "endogenous" variables (x's),
- r parameters (θ's).

The scalar condition $F_x \neq 0$ becomes invertibility of the endogenous Jacobian.

Implicit function theorem

Theorem 1.1 (Implicit function theorem)

Let F be C^1, let $F(x^0;\theta^0)=0$, and let $\mathrm{D}_x F(x^0;\theta^0)$ be invertible.

Then there are neighborhoods $\mathcal{U}$ of θ^0 and $\mathcal{V}$ of x^0 and a unique C^1 function $x:\mathcal{U}\to\mathcal{V}$ such that

$$x(\theta^0)=x^0 \qquad \text{and} \qquad F\big(x(\theta);\theta\big)=0 \quad \text{for every } \theta\in\mathcal{U},$$

and for each $\theta\in\mathcal{U}$ the point $x(\theta)$ is the only solution of $F(x;\theta)=0$ in $\mathcal{V}$.

The derivative of the branch satisfies

$$\mathrm{D}_x F\big(x(\theta);\theta\big)\,\mathrm{D}_\theta x(\theta)=-\mathrm{D}_\theta F\big(x(\theta);\theta\big).$$

Reading the matrix form of IFT

$$x = (x_1, \ldots, x_n)^\top, \qquad \theta = (\theta_1, \ldots, \theta_r)^\top, \qquad F = (F_1, \ldots, F_n)^\top.$$

Vectors are columns; Jacobian rows are outputs and columns are inputs.

$$\mathrm{D}_x F = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}_{n\times n}, \qquad \mathrm{D}_\theta x = \begin{pmatrix} \frac{\partial x_1}{\partial \theta_1} & \cdots & \frac{\partial x_1}{\partial \theta_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial \theta_1} & \cdots & \frac{\partial x_n}{\partial \theta_r} \end{pmatrix}_{n\times r},$$

$$\mathrm{D}_\theta F = \begin{pmatrix} \frac{\partial F_1}{\partial \theta_1} & \cdots & \frac{\partial F_1}{\partial \theta_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial \theta_1} & \cdots & \frac{\partial F_n}{\partial \theta_r} \end{pmatrix}_{n\times r}.$$

Evaluation: $\mathrm{D}_x F$ and $\mathrm{D}_\theta F$ at $(x(\theta); \theta)$; $\mathrm{D}_\theta x$ at θ.

What the implicit function theorem gives

Start from a known solution $F(x^0; \theta^0) = 0$ where $\mathrm{D}_x F(x^0; \theta^0)$ is invertible.

1. For every θ near θ^0, there is exactly one solution $x(\theta)$ near x^0.
    - Other solutions farther from x^0 may still exist.
2. To find the response to parameter θ_k, solve the $n \times n$ linear system
$$\mathrm{D}_x F \, \frac{\partial x}{\partial \theta_k} = -\frac{\partial F}{\partial \theta_k}.$$
3. Repeat for $k = 1, \ldots, r$ and stack the solution vectors as the columns of $\mathrm{D}_\theta x$:
$$\mathrm{D}_\theta x = -(\mathrm{D}_x F)^{-1} \mathrm{D}_\theta F.$$

Evaluation: derivatives of F at $(x(\theta); \theta)$; derivatives of x at θ.

Incidence of a per-unit tax

Example 1.2 (Incidence of a per-unit tax)

Fix demand and supply primitives $a, c \in \mathbb{R}$ and $b, s > 0$, and let the parameter be the per-unit tax t.

The endogenous variables are the producer price p and quantity q; consumers pay $p + t$.

Demand and supply are

$$q = a - b\,(p + t), \qquad q = c + s\,p.$$

At a reference tax t^0, suppose an equilibrium $x^0 = (p^0, q^0)^\top$ exists.

How do the local equilibrium price $p(t)$, quantity $q(t)$, and consumer price $p(t) + t$ respond to the tax?

Incidence of a per-unit tax: solution

Example 1.2 (Incidence of a per-unit tax)

With $b, s > 0$, equilibrium solves $q = a - b(p + t)$ and $q = c + s\,p$; find p_t, q_t, and $\mathrm{d}(p + t)/\mathrm{d}t$.

Incidence of a per-unit tax: solution

Example 1.2 (Incidence of a per-unit tax)

With $b, s > 0$, equilibrium solves $q = a - b(p+t)$ and $q = c + s\,p$; find p_t, q_t, and $\mathrm{d}(p+t)/\mathrm{d}t$.

Solution.

With $x = (p, q)^\top$, equilibrium is the system $F(p, q; t) = 0$, where

$$F = \begin{pmatrix} q - a + b\,(p+t) \\ q - c - s\,p \end{pmatrix}, \qquad \mathrm{D}_{(p,q)}F = \begin{pmatrix} b & 1 \\ -s & 1 \end{pmatrix}, \qquad \det \mathrm{D}_{(p,q)}F = b + s > 0,$$

so the theorem applies at any equilibrium, and differentiating in t,

$$\begin{pmatrix} b & 1 \\ -s & 1 \end{pmatrix} \begin{pmatrix} p_t \\ q_t \end{pmatrix} = \begin{pmatrix} -b \\ 0 \end{pmatrix} \implies p_t = -\frac{b}{b+s}, \qquad q_t = -\frac{bs}{b+s}.$$

The consumer price moves by $\mathrm{d}(p+t)/\mathrm{d}t = 1 + p_t = s/(b+s) \in (0,1)$. $\square$

Comparative statics of optimizers and values

Differentiating a first-order condition

Let $f : X \times \Theta \to \mathbb{R}$ be C^2 with $X \subseteq \mathbb{R}^n$ and $\Theta \subseteq \mathbb{R}^r$ open.

An interior optimizer x^0 at θ^0 satisfies Lecture 6's first-order condition

$$\nabla_x f(x^0; \theta^0) = 0.$$

Set $F(x; \theta) = \nabla_x f(x; \theta)$. Its endogenous Jacobian is the Hessian:

$$\mathrm{D}_x F(x; \theta) = H_f(x; \theta).$$

If $H_f(x^0; \theta^0)$ is invertible, the implicit function theorem gives a unique local C^1 branch of critical points satisfying

$$H_f\big(x(\theta); \theta\big)\, \mathrm{D}_\theta x(\theta) = -\mathrm{D}_\theta \, \nabla_x f\big(x(\theta); \theta\big).$$

Here we work with *critical points*, which may or may not be *optimizers*.

Reading the comparative-statics equation

Here the gradient $\nabla_x f$ is an $n \times 1$ column, so $H_f = \mathrm{D}_x(\nabla_x f)$ is $n \times n$:

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}_{n\times n}, \qquad \mathrm{D}_\theta x = \begin{pmatrix} \frac{\partial x_1}{\partial \theta_1} & \cdots & \frac{\partial x_1}{\partial \theta_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial \theta_1} & \cdots & \frac{\partial x_n}{\partial \theta_r} \end{pmatrix}_{n\times r},$$

$$\mathrm{D}_\theta \nabla_x f = \begin{pmatrix} \frac{\partial^2 f}{\partial \theta_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial \theta_r \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial \theta_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial \theta_r \partial x_n} \end{pmatrix}_{n\times r}.$$

Evaluation: H_f and $\mathrm{D}_\theta \nabla_x f$ at $(x(\theta);\theta)$; $\mathrm{D}_\theta x$ at θ.

The Lagrange system

Let $h : X \times \Theta \to \mathbb{R}^m$ be C^2 and let $\mu \in \mathbb{R}^m$. For

$$\max_x f(x;\theta) \qquad \text{subject to} \qquad h(x;\theta) = 0,$$

Lecture 7's Lagrangian is $L(x,\mu;\theta) = f(x;\theta) + \mu \cdot h(x;\theta)$.

Stationarity and feasibility form one system of $n+m$ equations:

$$G(x,\mu;\theta) = \begin{pmatrix} \nabla_x L(x,\mu;\theta) \\ h(x;\theta) \end{pmatrix} = 0$$

in the $n+m$ endogenous variables (x,μ).

The endogenous Jacobian in blocks

$$\mathrm{D}_{(x,\mu)} G = \begin{pmatrix} H_L & \mathrm{D}_x h^\top \\ \mathrm{D}_x h & 0 \end{pmatrix},$$

where H_L differentiates L in x alone. The block dimensions are

$$\begin{array}{c|cc} & x\ (n) & \mu\ (m) \\ \hline \nabla_x L\ (n) & n \times n & n \times m \\ h\ (m) & m \times n & m \times m \end{array}.$$

If this matrix is invertible at a stationary feasible pair, the implicit function theorem gives a unique local differentiable branch $(x(\theta), \mu(\theta))$ of such pairs.

Once these points are known to be optimizers, write the branch $(x^*(\theta), \mu^*(\theta))$.

For inequalities, hold the active set fixed

Let I be the same set of binding inequalities throughout a parameter region.

- Add $g_I(x;\theta) = 0$ and the corresponding multipliers λ_I to the Lagrange system.
- Omit the slack constraints; their multipliers are zero.
- Differentiate this equality system exactly as above.

These derivatives are valid only while the active set remains I; a switch changes the system being differentiated.

Envelope theorem: interior optimizers

Theorem 2.1 (Envelope theorems)

Assume the functions appearing below are C^1 in (x, θ).

Let $x^*(\theta)$ be a differentiable branch of interior optimizers and let $V(\theta) = f(x^*(\theta); \theta)$. Since $\nabla_x f(x^*(\theta); \theta) = 0$,

$$\mathrm{D}_\theta V(\theta) = \mathrm{D}_\theta f\left(x^*(\theta); \theta\right).$$

Envelope theorem: interior optimizers

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Let $x^*(\theta)$ be a differentiable branch of interior optimizers and let $V(\theta) = f(x^*(\theta); \theta)$. Since $\nabla_x f(x^*(\theta); \theta) = 0$,

$$\mathrm{D}_\theta V(\theta) = \mathrm{D}_\theta f\big(x^*(\theta); \theta\big).$$

Proof.

The chain rule gives

$$\mathrm{D}_\theta V = \nabla_x f^\top \mathrm{D}_\theta x^* + \mathrm{D}_\theta f,$$

and the first term vanishes by stationarity. $\square$

Envelope theorem: equality constraints

Theorem 2.1 (Envelope theorems)

Let $x^*(\theta)$ be a differentiable branch of optimizers and $\mu^*(\theta)$ its differentiable multiplier branch satisfying Lecture 7's stationarity and feasibility conditions. Then

$$\mathrm{D}_\theta V(\theta) = \mathrm{D}_\theta L\big(x^*(\theta), \mu^*(\theta); \theta\big) = \mathrm{D}_\theta f + \mu^*(\theta)^\top \mathrm{D}_\theta h,$$

with the derivatives on the right taken in θ alone and evaluated along the branch.

Envelope theorem: equality constraints

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with the derivatives on the right taken in θ alone and evaluated along the branch.

Proof.

The chain rule step and stationarity $\nabla_x f^\top = -(\mu^*)^\top \mathrm{D}_x h$ give

$$\mathrm{D}_\theta V = \mathrm{D}_\theta f + \nabla_x f^\top \mathrm{D}_\theta x^* = \mathrm{D}_\theta f - (\mu^*)^\top \mathrm{D}_x h \, \mathrm{D}_\theta x^*,$$

and differentiating $h(x^*(\theta); \theta) = 0$ gives $\mathrm{D}_x h \, \mathrm{D}_\theta x^* = -\mathrm{D}_\theta h$. □

Envelope theorem: inequality constraints

Theorem 2.1 (Envelope theorems)

For a problem with $h(x;\theta) = 0$ and $g(x;\theta) \geq 0$, suppose $x^*(\theta)$ is a differentiable branch of optimizers and $\mu^*(\theta)$, $\lambda^*(\theta)$ are differentiable KKT multiplier branches. If the active set is constant nearby, then

$$\mathrm{D}_\theta V(\theta) = \mathrm{D}_\theta L\big(x^*(\theta), \mu^*(\theta), \lambda^*(\theta); \theta\big) = \mathrm{D}_\theta f + \mu^*(\theta)^\top \mathrm{D}_\theta h + \lambda^*(\theta)^\top \mathrm{D}_\theta g.$$

The envelope formula on the quadratic family

Example 2.2 (The envelope formula on the quadratic family)

For each parameter $\theta \in \mathbb{R}$, the scalar choice variable x solves $\max_{x \geq 0} \; \theta x - \frac{1}{2}x^2$.

Recall from Lecture 6 that the optimizer and value are

$$x^*(\theta) = \max\{\theta, 0\}, \qquad V(\theta) = \begin{cases} \frac{1}{2}\theta^2, & \theta > 0, \\ 0, & \theta \leq 0. \end{cases}$$

Calculate $V'(\theta)$.

The envelope formula on the quadratic family: solution

Example 2.2 (The envelope formula on the quadratic family)

For the quadratic family, calculate $V'(\theta)$ on $\theta > 0$, on $\theta < 0$, and at $\theta = 0$.

The envelope formula on the quadratic family: solution

Example 2.2 (The envelope formula on the quadratic family)

For the quadratic family, calculate $V'(\theta)$ on $\theta > 0$, on $\theta < 0$, and at $\theta = 0$.

Solution.

On the interior regime $\theta > 0$, part 1 predicts

$$V'(\theta) = \frac{\partial}{\partial\theta}\left(\theta x - \tfrac{1}{2}x^2\right)\Big|_{x=x^*(\theta)} = x^*(\theta) = \theta,$$

matching the derivative of $\frac{1}{2}\theta^2$. On the corner regime $\theta < 0$, write the constraint as $g(x) = x \geq 0$: the active set is constant, $x^* = 0$, and part 3 gives $V'(\theta) = \partial f/\partial\theta = x^*(\theta) = 0$.

At $\theta = 0$, the active set changes, but the displayed value function has matching one-sided derivatives, so $V'(0) = 0$. □

Put a parameter on the constraint

Start from the equality-constrained problem

$$\max_x f(x) \qquad \text{subject to} \qquad h(x) = 0.$$

Replace the fixed right-hand side 0 by a scalar parameter b:

$$\max_x f(x) \qquad \text{subject to} \qquad h(x) = b.$$

Let $x^*(b)$ be an optimizer and let

$$V(b) = f\left(x^*(b)\right)$$

be the best attainable value. The original problem is $b = 0$.

Changing b moves the constraint; $V'(b)$ measures the resulting marginal change in the optimized value.

The multiplier is the value of moving the constraint

Write $h(x) = b$ in the course's sign convention:

$$b - h(x) = 0, \qquad L(x, \mu; b) = f(x) + \mu\big(b - h(x)\big).$$

The equality-constrained envelope formula gives

$$\boxed{V'(b) = \frac{\partial L}{\partial b}\big(x^*(b), \mu^*(b); b\big) = \mu^*(b)}.$$

Thus the multiplier is the *shadow price* of the constraint's right-hand side:

$$V(b + \Delta b) - V(b) \approx \mu^*(b)\,\Delta b \qquad \text{for small } \Delta b.$$

At the original problem, $\mu^*(0)$ is the marginal value of moving the right-hand side away from zero.

The same interpretation for an inequality

For an upper bound $r(x) \leq b$, use Lecture 7's standard form

$$g(x; b) = b - r(x) \geq 0, \qquad L(x, \lambda; b) = f(x) + \lambda g(x; b), \qquad \lambda \geq 0.$$

On a differentiable regime with a fixed active set, the inequality envelope formula gives

$$\boxed{V'(b) = \frac{\partial L}{\partial b} = \lambda^*(b)}.$$

- If $g(x^*(b); b) = 0$, then $\lambda^*(b)$ is the marginal value of increasing b.
- If $g(x^*(b); b) > 0$, complementary slackness gives $\lambda^*(b) = 0$.

Capacity: read the derivatives from L

Example 2.3 (Capacity valued by the envelope formula)

For $a, K > 0$, let

$$V(a, K) = \max_{0 \le q \le K} \left\{ aq - \tfrac{1}{2} q^2 \right\}.$$

With $g_2(q; K) = K - q$, Lecture 7 gives $q^* = \min\{a, K\}$ and $\lambda_2^* = \max\{a - K, 0\}$.
For $a \neq K$, find V_K and V_a without solving for V.

Capacity: read the derivatives from L

Example 2.3 (Capacity valued by the envelope formula)

For $a, K > 0$, let

$$V(a, K) = \max_{0 \le q \le K} \left\{ aq - \tfrac{1}{2} q^2 \right\}.$$

With $g_2(q; K) = K - q$, Lecture 7 gives $q^* = \min\{a, K\}$ and $\lambda_2^* = \max\{a - K, 0\}$. For $a \neq K$, find V_K and V_a without solving for V.

Solution.

On each regime, K enters L through $\lambda_2(K - q)$ and a through aq. Hence

$$V_K = \lambda_2^* = \max\{a - K, 0\}, \qquad V_a = q^* = \min\{a, K\}.$$

At $a = K$ the constraint binds with $\lambda_2^* = 0$; the theorem does not cover the active-set switch. $\square$

Example 2.4 (Hotelling's lemma)

A firm chooses an input vector $x \in \mathbb{R}^n$ and produces output $y(x)$.

The parameters are the output price p and input-price vector $w \in \mathbb{R}^n$, and profit is

$$\pi(x; p, w) = p\, y(x) - w \cdot x.$$

Suppose $x^*(p, w)$ is a differentiable branch of interior profit maximizers, with maximized profit $\Pi(p, w)$.

What are the derivatives of Π with respect to p and each w_i?

Hotelling's lemma: solution

Example 2.4 (Hotelling's lemma)

With $\pi(x; p, w) = p\, y(x) - w \cdot x$ maximized at an interior branch $x^*(p, w)$, find $\partial\Pi/\partial p$ and $\partial\Pi/\partial w_i$.

Hotelling's lemma: solution

Example 2.4 (Hotelling's lemma)

With $\pi(x; p, w) = p\,y(x) - w \cdot x$ maximized at an interior branch $x^*(p, w)$, find $\partial\Pi/\partial p$ and $\partial\Pi/\partial w_i$.

Solution.

Part 1 of the envelope theorem gives

$$\frac{\partial \Pi}{\partial p} = y\big(x^*(p, w)\big), \qquad \frac{\partial \Pi}{\partial w_i} = -x_i^*(p, w).$$

The same envelope logic underlies Roy's identity and Shephard's lemma. □

Fixed-point theorems

Fixed point

Definition 3.1 (Fixed point)

Let X be a set and $f : X \to X$.

A point $x^* \in X$ is a *fixed point* of f if $f(x^*) = x^*$.

In one variable, a fixed point is where the graph of f meets the 45-degree line.

Lecture 2 closed with our first fixed point theorem, Brouwer in $\mathbb{R}$.

- Every continuous $f : [a, b] \to [a, b]$ has a fixed point.

A continuous self-map of an interval meets the diagonal

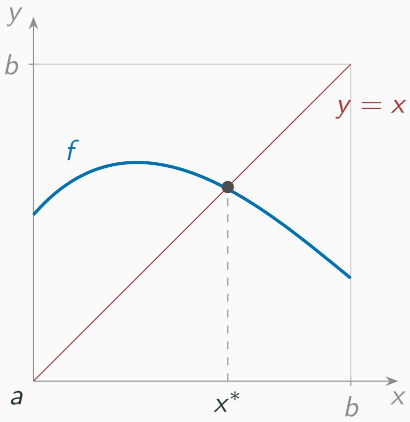

Figure 3.1

Brouwer

In $\mathbb{R}^n$ the interval is replaced by a compact convex set.

Theorem 3.2 (Brouwer)

Let $D \subseteq \mathbb{R}^n$ be nonempty, compact, and convex, and let $f : D \to D$ be continuous.

Then f has a fixed point.

Set-valued map

Definition 3.3 (Set-valued map)

A *set-valued map* (*correspondence*) $\Gamma : X \rightrightarrows Y$ assigns to every $x \in X$ a nonempty set $\Gamma(x) \subseteq Y$.

A *fixed point* of $\Gamma : X \rightrightarrows X$ is a point x^* with $x^* \in \Gamma(x^*)$.

For correspondences, Kakutani uses a closed-graph condition.

- The *graph* of Γ is the set of pairs $\{(x, y) : x \in X,\ y \in \Gamma(x)\}$.
- It must be a closed set, in the sequential sense of Lecture 2.

In words, if $x_k \to x$, $y_k \to y$, and $y_k \in \Gamma(x_k)$ for every k, then $y \in \Gamma(x)$.

Theorem 3.4 (Kakutani)

Let $D \subseteq \mathbb{R}^n$ be nonempty, compact, and convex.

Let $\Gamma : D \rightrightarrows D$ have convex values and a closed graph.

Then Γ has a fixed point.

Intuition:

- A closed graph preserves limiting choices, and convex values fill in between them.

Why the closed-graph condition matters

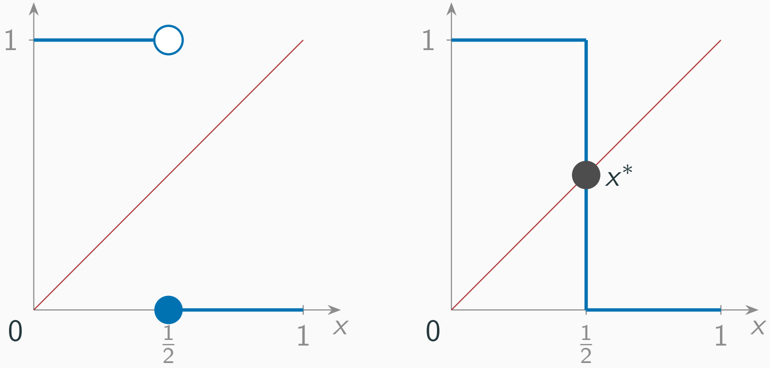

Figure 3.2

- The nonclosed graph skips the diagonal; the convex tie value restores a fixed point.

Contraction

The next result imposes a stronger condition than continuity but yields a unique fixed point and an algorithm for calculating it.

Definition 3.6 (Contraction)

Let $S \subseteq \mathbb{R}^n$.

A map $f : S \to S$ is a *contraction with modulus* β if $0 \leq \beta < 1$ and

$$\|f(x) - f(y)\| \leq \beta \|x - y\| \qquad \text{for all } x, y \in S.$$

A contraction shrinks the distance between any two points.

- The new distance is at most β times the original distance.

Every contraction is continuous: if $x_k \to x$, then $\|f(x_k) - f(x)\| \leq \beta\|x_k - x\| \to 0$.

Contraction mapping theorem

Theorem 3.7 (Contraction mapping theorem)

Let $S \subseteq \mathbb{R}^n$ be nonempty and closed.

Let $f : S \to S$ be a contraction with modulus β.

Then f has exactly one fixed point $x^* \in S$.

Moreover, take any starting point $x_0 \in S$.

The iterates $x_{k+1} = f(x_k)$ converge to x^*, and

$$\|x_k - x^*\| \leq \beta^k \|x_0 - x^*\|.$$

Here S need not be bounded or convex.

- The strong hypothesis on f replaces both.

Why the fixed point is unique

Remark 3.8 (Why the fixed point is unique)

A contraction moves every pair of distinct points closer together.

A fixed point does not move.

- If two distinct fixed points existed, applying f would have to leave their distance unchanged and shrink it at the same time.
- Therefore there can be only one.

Iteration has the same intuition: each application of f reduces the remaining distance to the fixed point by a factor of at most β.

- The errors shrink geometrically, so the iterates are pulled toward the fixed point.

A discounted sum as a fixed point

Example 3.9 (A discounted sum as a fixed point)

Fix parameters $a \in \mathbb{R}$ and $\beta \in (0, 1)$.

The fixed-point variable is $x \in S = \mathbb{R}$, and the map is $T : S \to S$ defined by $T(x) = a + \beta x$.

Does T have a unique fixed point, and what are the iterates $x_{k+1} = T(x_k)$ from $x_0 = 0$?

A discounted sum as a fixed point: solution

Example 3.9 (A discounted sum as a fixed point)

For $T(x) = a + \beta x$ on $S = \mathbb{R}$ with $\beta \in (0, 1)$, find the fixed point and the iterates from $x_0 = 0$.

A discounted sum as a fixed point: solution

Example 3.9 (A discounted sum as a fixed point)

For $T(x) = a + \beta x$ on $S = \mathbb{R}$ with $\beta \in (0, 1)$, find the fixed point and the iterates from $x_0 = 0$.

Solution.

Since $|T(x) - T(y)| = \beta|x - y|$, the map is a contraction. Its unique fixed point is $x^* = a/(1 - \beta)$.

Starting the iteration at $x_0 = 0$, for $k \geq 1$,

$$x_k = a\left(1 + \beta + \cdots + \beta^{k-1}\right) = a\,\frac{1 - \beta^k}{1 - \beta} \longrightarrow \frac{a}{1 - \beta} = x^*.$$

This is the value of receiving a in every period, discounted by β. Lecture 9 replaces x by a function. $\square$

The fixed-point theorems used in the first-year sequence

Theorem	The set	The map	Conclusion
Brouwer	nonempty, compact, convex	continuous $f : D \to D$	a fixed point exists
Kakutani	nonempty, compact, convex	$\Gamma : D \rightrightarrows D$; convex values, closed graph	a fixed point exists
Contraction	nonempty, closed	contraction $f : S \to S$ with modulus $\beta < 1$	unique fixed point; iteration converges

Table 3.1