Constrained Optimization

Lecture 7

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Econ 8000 — Math Camp

Overview: constraints, tangent directions, and multipliers

Lecture 6's method and its boundary condition

Lecture 6 solved an optimization problem in three separate steps.

- Establish that a solution exists.
- Generate candidates from first-order conditions.
- Certify a candidate as global, by concavity or by comparing values.

Its first-order condition at a boundary point was one-sided: at a local maximizer x^*,

$$\nabla f(x^*) \cdot v \leq 0 \qquad \text{for every feasible direction } v \text{ from } x^*. \tag{FOC}$$

v is a feasible direction if $x^* + tv$ stays in the feasible set for all small $t > 0$.

Constraints in economic problems

In Economics, the feasible set is typically described by equations and inequalities:

- A consumer's budget constraint $p \cdot x \leq m$ is a linear inequality.
- A planner's constraint $x_1 + x_2 = m$ is a linear equality.
  - It says that a resource must be split between two uses.
- A firm's production constraint is $q = F(z)$.
  - It links output q to inputs z.
  - It is a nonlinear equality whenever the production function F is nonlinear.

The affine case: (FOC) has content

Depending on the shape of the feasible set, condition (FOC) may be vacuous.

Suppose that there is just one *equality* constraint $h(x) = 0$.

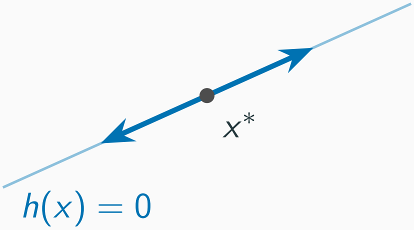
If h is affine, as in the planner's problem, the feasible set is a line.

- Segments along the line stay in the set.
- So nonzero feasible directions exist and (FOC) is not vacuous.

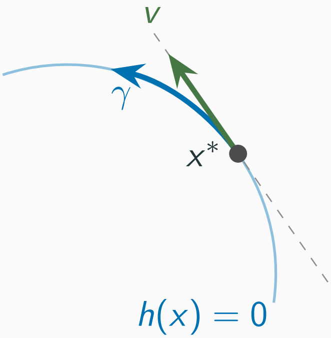
A curved feasible set, such as the circle $x_1^2 + x_2^2 = 2$, contains no segments.

- Every straight move away from a point of the circle leaves the circle.
- So (FOC) holds at every point of the circle and rules out nothing.

Feasible movement under an equality constraint



linear constraint



curved constraint

Plan for Lecture 7

Part 1: Equality constraints and the Lagrange multiplier method.

Part 2: Inequality constraints and the Karush–Kuhn–Tucker (KKT) conditions.

Part 3: Second-order conditions and global certification.

Equality constraints and the Lagrangian

Maximizing subject to equations

Let $U \subseteq \mathbb{R}^n$ be open, let $f : U \to \mathbb{R}$ and $h : U \to \mathbb{R}^k$ be C^1, and consider

$$\max_{x \in U} f(x) \qquad \text{subject to} \qquad h(x) = 0.$$

The feasible set $\{x \in U : h(x) = 0\}$ is described by a system of k equations.

When one of these equations is nonlinear, the feasible set may be curved.

- We thus work with feasible curves and their velocities.

Feasible curves and tangent directions

A differentiable γ is a *feasible curve through* x^* if $\gamma(0) = x^*$ and $h(\gamma(t)) = 0$ near 0.

Its velocity at x^* is $v = \gamma'(0)$.

- Such a velocity is called a *tangent direction* to the constraint set at x^*.

Tangent directions lie in the null space

Lemma 2.1 (Tangent directions lie in the null space)

Let γ be a feasible curve through x^* with velocity v.

Then $\mathrm{D}h(x^*)\, v = 0$.

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Let γ be a feasible curve through x^* with velocity v.

Then $\mathrm{D}h(x^*)\,v = 0$.

Proof.

$h(\gamma(t)) = 0$ near 0, and differentiating using the chain rule from Lecture 4 gives

$$\mathrm{D}h(\gamma(t))\,\gamma'(t) = 0 \quad \text{at } t = 0, \quad \text{so} \quad \mathrm{D}h(x^*)\,v = 0.$$

$\square$

Normals to the constraint set

Every tangent direction thus lies in the null space $N(\mathrm{D}h(x^*))$ from Lecture 3.

Row by row, the lemma says

$$\nabla h_j(x^*) \cdot v = 0 \qquad \text{for each } j.$$

Each constraint gradient is thus *orthogonal* (perpendicular) to every tangent direction.

A vector perpendicular to the tangent directions is a *normal* to the constraint set.

- Thus the constraint gradients are normals to the constraint set.

The local geometry at x^* for one constraint ($k = 1$)

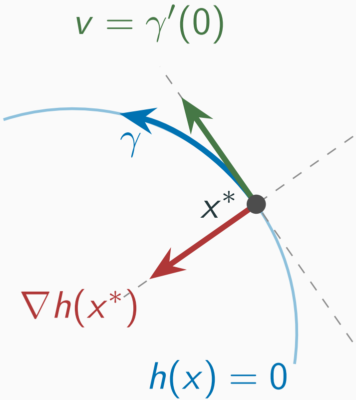

Figure 2.1

The converse is generally not true

If $\mathrm{D}h(x^*)\, v = 0$, there may be no feasible curve through x^* with velocity v.

Example: If $h(x) = x^2$, the feasible set is the single point $\{0\}$.

- The only feasible velocity at 0 is 0.
- The null space of $h'(0) = 0$ is all of $\mathbb{R}$, so any vector $v \in \mathbb{R}$ satisfies $\mathrm{D}h(0)\, v = 0$.

LICQ for equality constraints

The following condition ensures that all vectors in the null space of $\mathrm{D}h(x^*)$ are velocities of feasible curves through x^*.

Definition 2.2 (LICQ for equality constraints)

The *linear independence constraint qualification* holds at a feasible point x^* if the constraint gradients

$$\nabla h_1(x^*), \ldots, \nabla h_k(x^*)$$

are linearly independent; equivalently, if $\operatorname{rank} \mathrm{D}h(x^*) = k$.

Lemma 2.4 (Tangent directions under LICQ)

Let h be C^1 and let LICQ hold at the feasible point x^*.

If $\mathrm{D}h(x^*)\,v = 0$, then v is the velocity of a differentiable feasible curve through x^*.

Intuitively, a vector v orthogonal to the normals is tangent to the constraint set.

- A curve can then be drawn along the constraint set in the direction of v.

LICQ thus ensures that the constraints form a smooth surface locally around x^*.

Combining the lemmas: under LICQ, the feasible velocities are exactly $N(\mathrm{D}h(x^*))$.

The first-order condition over the null space

Now suppose x^* is a local maximizer at which LICQ holds.

For any feasible curve through x^*, $t \mapsto f(\gamma(t))$ has a local maximum at $t = 0$.

- The point $t = 0$ is interior, so the derivative at $t = 0$ is zero.

The chain rule and the lemma on tangent directions under LICQ thus yield

$$\nabla f(x^*) \cdot v = 0 \qquad \text{for every } v \in N(\mathrm{D}h(x^*)).$$

This is the equality-constraint analogue of condition (FOC).

The condition holds with equality

Simply put, $\nabla f(x^*)$ is orthogonal to every tangent direction.

- So it must be a linear combination of the constraint gradients:

$$\nabla f(x^*) + \mu_1^* \nabla h_1(x^*) + \cdots + \mu_k^* \nabla h_k(x^*) = 0.$$

These coefficients are called the *Lagrange multipliers*.

Lagrange multiplier theorem

Theorem 2.5 (Lagrange multiplier theorem)

Let f and h be C^1.

Let x^* be a local extremum (maximizer or minimizer) of f subject to $h(x) = 0$.

Let LICQ hold at x^*.

Then there is a unique vector $\mu^* \in \mathbb{R}^k$ such that

$$\nabla f(x^*) + \mathrm{D}h(x^*)^\top \mu^* = 0.$$

Reading the multiplier theorem

As in Lecture 6, the theorem gives only a necessary condition for a local extremum.

Note that the Lagrange multipliers are not restricted in sign.

- They can be positive or negative.

For a fixed objective, constraint description, and candidate x^*, LICQ makes μ^* unique.

- Rescaling or rewriting a constraint (e.g., multiplying by 2) changes its multiplier.

Finally, the theorem does not guarantee that an optimizer exists.

- Existence must be established separately.

LICQ cannot be dropped

LICQ cannot be dropped from the multiplier theorem.

- All points in the feasible set that do not satisfy LICQ must be treated separately.

A singular description loses the multiplier

Example 2.6 (A singular description loses the multiplier)

Consider $\max_{x \in \mathbb{R}} x$ subject to $h(x) = x^2 = 0$.

Find the global maximizer and a multiplier μ for it.

A singular description loses the multiplier

Example 2.6 (A singular description loses the multiplier)

Consider $\max_{x \in \mathbb{R}} x$ subject to $h(x) = x^2 = 0$.

Find the global maximizer and a multiplier μ for it.

Solution.

We have $f'(x) = 1$ and $h'(x) = 2x$.

Applying the Lagrange multiplier theorem would give $1 + 2x\mu = 0$.

However, the feasible set is $\{0\}$, meaning that $x^* = 0$ is the global maximizer.

Clearly, at $x^* = 0$ no μ satisfies the multiplier condition $1 + 2x^*\mu = 0$. $\square$

The multiplier condition and feasibility $h(x) = 0$ form a system of $n + k$ equations.

- The unknowns are the $n + k$ coordinates of (x, μ).

They can be written as the critical-point conditions of one function of $n + k$ variables.

Definition 2.7 (Lagrangian)

The *Lagrangian* of the problem $\max f(x)$ subject to $h(x) = 0$ is the function of x and μ given by

$$L(x, \mu) = f(x) + \mu \cdot h(x).$$

The Lagrangian collects the first-order conditions

Its partial gradients are:

$$\nabla_x L(x, \mu) = \nabla f(x) + \mathrm{D}h(x)^\top \mu, \qquad \nabla_\mu L(x, \mu) = h(x).$$

The critical points of L are thus exactly the pairs satisfying stationarity and feasibility.

We do <u>not</u> maximize the Lagrangian.

- A constrained maximizer is a critical point of $L(\,\cdot\,, \mu^*)$.
- It can thus be a maximum, a minimum, or a saddle point of the Lagrangian.

Solving directly

Example 2.8 (Solving directly and with a Lagrangian)

Consider max $x_1 x_2$ subject to $x_1 + x_2 = 2$. Solve it directly, then with a Lagrangian.

Solving directly

Example 2.8 (Solving directly and with a Lagrangian)

Consider $\max\, x_1 x_2$ subject to $x_1 + x_2 = 2$. Solve it directly, then with a Lagrangian.

Solution.

Direct solution. The constraint gives $x_2 = 2 - x_1$, so the problem reduces to

$$\max_{x_1 \in \mathbb{R}} \; x_1(2 - x_1) = \max_{x_1 \in \mathbb{R}} \left[1 - (x_1 - 1)^2\right].$$

The unique global maximizer is $x_1^* = 1$, and the constraint then gives $x_2^* = 1$.

- Thus $x^* = (1, 1)^\top$.

$\square$

Solving with a Lagrangian

Example 2.8 (Solving directly and with a Lagrangian)

Consider max $x_1 x_2$ subject to $x_1 + x_2 = 2$. Solve it directly, then with a Lagrangian.

Solution.

Lagrangian solution. With $h(x) = 2 - x_1 - x_2$, $\nabla h(x) = (-1, -1)^\top \neq 0$, so LICQ holds, and stationarity of the Lagrangian $L(x, \mu) = x_1 x_2 + \mu\,(2 - x_1 - x_2)$ requires

$$x_2 - \mu = 0, \qquad x_1 - \mu = 0, \qquad 2 - x_1 - x_2 = 0.$$

The first two equations give $x_1 = x_2 = \mu$; the third then gives $\mu^* = 1$ and $x^* = (1, 1)^\top$.

- The same point as above, but identified only as a candidate.

□

A saddle point of the Lagrangian

Example 2.8 (Solving directly and with a Lagrangian)

Consider max $x_1 x_2$ subject to $x_1 + x_2 = 2$. Solve it directly, then with a Lagrangian.

Solution.

As a function of x with $\mu = \mu^* = 1$ fixed, $L(x, \mu^*) = x_1 x_2 + 2 - x_1 - x_2$ has Hessian $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, with eigenvalues 1 and -1.

- Indefinite, so x^* is a saddle point of $L(\,\cdot\,, \mu^*)$ by the classification of Lecture 6.
- On the feasible set, $h(x) = 0$ makes $L(x, \mu^*) = f(x)$: x^* maximizes L among feasible points.
- The Lagrangian does not remove the constraint.

$\square$

Inequalities and the KKT conditions

Standard form for inequalities

We fix one convention for inequality constraints.

Every inequality is written as $g_i(x) \geq 0$.

- So $a(x) \leq b$ becomes $b - a(x) \geq 0$.

The optimization problem is

$$\max_{x \in U} f(x) \qquad \text{subject to} \qquad h(x) = 0, \quad g(x) \geq 0,$$

where $h : U \to \mathbb{R}^k$ and $g : U \to \mathbb{R}^\ell$.

The Lagrangian of the mixed problem

Its Lagrangian is

$$L(x,\mu,\lambda) = f(x) + \mu \cdot h(x) + \lambda \cdot g(x), \qquad \mu \in \mathbb{R}^k, \quad \lambda \geq 0.$$

The requirement $\lambda \geq 0$ is forced by the convention $g \geq 0$.

Active and slack constraints, mixed LICQ

For inequality constraints, we will use the following constraint classification.

Definition 3.1 (Active and slack constraints, mixed LICQ)

At a feasible point x, the *active set* is

$$I(x) = \{i : g_i(x) = 0\},$$

and a constraint with $g_i(x) > 0$ is *slack*.

LICQ holds at x for the mixed problem if the vectors

$$\{\nabla h_j(x) : j = 1, \dots, k\} \cup \{\nabla g_i(x) : i \in I(x)\}$$

are linearly independent.

Slack constraints stay out of the qualification

A slack constraint stays satisfied throughout a neighborhood of x, by continuity of g_i.

- So it places no restriction on local movement.
- Its gradient does not enter the constraint qualification.

Theorem 3.2 (Karush–Kuhn–Tucker)

Let f, h, and g be C^1.

Let x^* be a local maximizer subject to $h(x) = 0$ and $g(x) \geq 0$.

Let LICQ hold at x^*.

Then there are $\mu^* \in \mathbb{R}^k$ and $\lambda^* \in \mathbb{R}^\ell$ such that

$$
\begin{aligned}
\nabla f(x^*) + \mathrm{D}h(x^*)^\top \mu^* + \mathrm{D}g(x^*)^\top \lambda^* &= 0 && \textit{(stationarity)} \\
h(x^*) = 0, \qquad g(x^*) &\geq 0 && \textit{(feasibility)} \\
\lambda^* &\geq 0 && \textit{(multiplier signs)} \\
\lambda_i^* g_i(x^*) &= 0 \quad (i = 1, \ldots, \ell) && \textit{(complementary slackness)}
\end{aligned}
$$

Reading complementary slackness

Complementary slackness forces $\lambda_i^* = 0$ whenever $g_i(x^*) > 0$.

- It permits $\lambda_i^* > 0$ only where $g_i(x^*) = 0$: slack constraints have zero multipliers.

The converse implication fails.

- An *active* constraint can also have a zero multiplier.

Why the multipliers are nonnegative

Here is the intuition why the multipliers λ^* are nonnegative.

Suppose $n = \ell = 1$, the constraint $g(x^*) = 0$ is active, and $g'(x^*) > 0$.

- So $g(x^* + t) > 0$ for every small $t > 0$, and $v = 1$ is a feasible direction.

If x^* is a local maximizer, the objective cannot increase in a feasible direction.

- This is condition (FOC) with $v = 1$: $f'(x^*) \leq 0$.

From stationarity, $f'(x^*) + \lambda^* g'(x^*) = 0$, so

$$\lambda^* = -\frac{f'(x^*)}{g'(x^*)} \geq 0.$$

The convention has to be recorded

With the convention $g \leq 0$, the same calculation would produce $\lambda^* \leq 0$.

- Thus the convention must be fixed before solving the problem.

KKT points still need certification

Like every first-order condition in this course, the KKT conditions generate candidates.

- They do not certify them (not even locally).

Existence and global certification remain separate steps.

Active-set reasoning

With ℓ inequalities, the practical method is to reason or guess which constraints bind.

Each active-set guess turns the KKT conditions into equalities that can be solved.

- Feasibility and the multiplier signs then confirm or reject the guess.

KKT by guess-and-verify

Example 3.3 (KKT by guess-and-verify)

Let $a > 0$ and $K > 0$. Consider $\max_q \, aq - \frac{1}{2}q^2$ subject to $q \geq 0$ and $K - q \geq 0$. Find q^* and the multipliers for every a and K.

KKT by guess-and-verify

Example 3.3 (KKT by guess-and-verify)

Let $a > 0$ and $K > 0$. Consider $\max_q aq - \frac{1}{2}q^2$ subject to $q \geq 0$ and $K - q \geq 0$. Find q^* and the multipliers for every a and K.

Solution.

The feasible set $[0, K]$ is nonempty and compact and the objective is continuous, so a maximizer exists by Weierstrass.

With $g_1(q) = q$ and $g_2(q) = K - q$, the Lagrangian is

$$L(q, \lambda) = aq - \tfrac{1}{2}q^2 + \lambda_1 q + \lambda_2(K - q),$$

and the KKT conditions are

$$a - q + \lambda_1 - \lambda_2 = 0, \qquad 0 \leq q \leq K, \qquad \lambda_1, \lambda_2 \geq 0, \qquad \lambda_1 q = 0, \quad \lambda_2(K - q) = 0. \quad \square$$

Checking the four cases

Example 3.3 (KKT by guess-and-verify)

Let $a > 0$ and $K > 0$. Consider $\max_q aq - \frac{1}{2}q^2$ subject to $q \geq 0$ and $K - q \geq 0$. Find q^* and the multipliers for every a and K.

Solution.

1. $\lambda_1 = \lambda_2 = 0$: stationarity gives $q = a$, feasible if and only if $K \geq a$.
2. $\lambda_1 = 0$, $K - q = 0$: stationarity gives $\lambda_2 = a - K$, admissible if and only if $K \leq a$.
3. $\lambda_2 = 0$, $q = 0$: stationarity gives $\lambda_1 = -a < 0$, violating $\lambda_1 \geq 0$ — rejected.
4. $\lambda_1, \lambda_2 > 0$: impossible, since $q = 0$ and $q = K$ cannot both hold for $K > 0$.

The solution is thus $q^*(a, K) = \min\{a, K\}$ with $\lambda_2^*(a, K) = \max\{a - K, 0\}$. □

Certification and a zero multiplier

Example 3.3 (KKT by guess-and-verify)

Let $a > 0$ and $K > 0$. Consider $\max_q \; aq - \frac{1}{2}q^2$ subject to $q \geq 0$ and $K - q \geq 0$. Find q^* and the multipliers for every a and K.

Solution.

To certify that $q^* = \min\{a, K\}$ is a global maximizer, rewrite the objective:

$$aq - \frac{1}{2}q^2 = -\frac{1}{2}(q - a)^2 + \frac{1}{2}a^2.$$

The objective is maximized by the point in $[0, K]$ closest to a, which is $\min\{a, K\} = q^*$.

- When $a = K$: $q^* = K$ with $\lambda_2^* = 0$ — an active constraint with a zero multiplier.

$\square$

Global certification

Concavity certifies again

In Lecture 6, if the objective f is concave on a convex set, then the first-order condition is sufficient for a global maximum.

- The same logic extends to an important class of constrained problems.

The certification again relies on the tangent inequality from Lecture 5,

$$f(y) \leq f(x) + \nabla f(x) \cdot (y - x).$$

Concave programs

Definition 4.1 (Concave program)

Let $U \subseteq \mathbb{R}^n$ be convex. The problem

$$\max_{x \in U} f(x) \qquad \text{subject to} \qquad h(x) = 0, \quad g(x) \geq 0$$

is a *concave program* if

- f and every g_i are concave on U, and
- h is affine.

Sufficiency in a concave program

Theorem 4.2 (Sufficiency in a concave program)

Consider a concave program on an open convex set U, and suppose that f and every g_i are differentiable.

Suppose a feasible x^* and multipliers (μ^*, λ^*) satisfy the KKT conditions.

Then x^* is a global maximizer.

Note that no constraint qualification is needed.

The proof (in the lecture notes, optional) uses the tangent inequality, stationarity, and complementary slackness.

Uniqueness in a concave program

Recall Lecture 6's concave maximization theorem (Theorem 3.1, part 3).

- A strictly concave function on a convex set has at most one maximizer.

The constraints of a concave program define a convex feasible set (possibly empty).

- The intersection of superlevel sets of concave functions and an affine equality set.
- Each of these sets is convex by Lecture 5.

Thus, if f is strictly concave, the program has at most one maximizer.

Slater's condition

KKT conditions are sufficient in a concave program.

Slater's condition gives a constraint qualification under which they are also necessary.

- It requires the set of strictly feasible points (which are points satisfying the equalities and every inequality strictly) to be nonempty.

Definition 4.3 (Slater's condition)

A concave program with affine equalities satisfies *Slater's condition* if there is $\bar{x} \in U$ with

$$h(\bar{x}) = 0 \qquad \text{and} \qquad g_i(\bar{x}) > 0 \quad \text{for every } i.$$

Necessity under Slater's condition

Theorem 4.4 (Necessity under Slater's condition)

Consider a concave program on an open convex set U, and suppose that f and every g_i are differentiable.

If Slater's condition holds, then every global maximizer satisfies the KKT conditions.

It satisfies them with some multipliers (μ^*, λ^*).

Slater and LICQ are different

Slater and LICQ are both constraint qualifications that can guarantee the existence of multipliers.

- LICQ is local and point-specific.
- Slater is global and specific to concave programs.

Slater does not require the active constraint gradients at the optimizer to be linearly independent.

A concave program with no multipliers

Example 4.5 (A concave program with no multipliers)

Consider $\max_{(x_1,x_2)\in\mathbb{R}^2} x_2$ subject to $x_1 \geq 0$ and $-x_1 - x_2^2 \geq 0$. Find the global maximizer and check the KKT conditions at it.

A concave program with no multipliers

Example 4.5 (A concave program with no multipliers)

Consider $\max_{(x_1,x_2)\in\mathbb{R}^2} x_2$ subject to $x_1 \geq 0$ and $-x_1 - x_2^2 \geq 0$. Find the global maximizer and check the KKT conditions at it.

Solution.

This is a concave program: the objective and $g_1(x) = x_1$ are affine, and $g_2(x) = -x_1 - x_2^2$ is concave.

Feasibility requires $0 \leq x_1 \leq -x_2^2 \leq 0$.

- So the feasible set is $\{(0,0)\}$ and $x^* = (0,0)$ is the unique global maximizer.
- Slater fails: $g_1(x) > 0$ requires $x_1 > 0$, while $g_2(x) > 0$ requires $x_1 < -x_2^2 \leq 0$.

$\square$

Stationarity has no solution

Example 4.5 (A concave program with no multipliers)

Consider $\max_{(x_1,x_2)\in\mathbb{R}^2} x_2$ subject to $x_1 \geq 0$ and $-x_1 - x_2^2 \geq 0$. Find the global maximizer and check the KKT conditions at it.

Solution.

At the global maximizer $x^* = (0,0)$, stationarity would require

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0.$$

- Impossible: the second component is always 1, so the global maximizer has no KKT multipliers.

□

Curvature along the tangent directions

For equality-constrained problems there is also a second-order local test.

The relevant matrix is the Hessian of the *Lagrangian*, not of the objective.

- Curvature only matters along the tangent directions.
- Along those directions both the objective and the constraints are curves.

The multiplier terms in L account for the curvature of the constraints.

Second-order conditions on the tangent space

Theorem 4.6 (Second-order conditions on the tangent space)

Let f and h be C^2, let x^* be feasible with LICQ, and let μ^* satisfy the stationarity condition from the Lagrange multiplier theorem:

$$\nabla f(x^*) + \mathrm{D}h(x^*)^\top \mu^* = 0.$$

Write

$$T = N(\mathrm{D}h(x^*)), \qquad H = H_f(x^*) + \sum_{j=1}^{k} \mu_j^* H_{h_j}(x^*),$$

so H is the Hessian of $L(\,\cdot\,, \mu^*)$ at x^*.

1. *If x^* is a local constrained maximizer, then $v^\top H v \le 0$ for every $v \in T$.*
2. *If $v^\top H v < 0$ on $T \setminus \{0\}$, then x^* is a strict local constrained maximizer.*

Example 4.7 (Classifying two candidates on a circle)

Consider $\max x_1 + x_2$ subject to $h(x) = 2 - x_1^2 - x_2^2 = 0$. Find the candidates, classify them, and certify the global maximizer.

Classifying two candidates on a circle

Example 4.7 (Classifying two candidates on a circle)

Consider max $x_1 + x_2$ subject to $h(x) = 2 - x_1^2 - x_2^2 = 0$. Find the candidates, classify them, and certify the global maximizer.

Solution.

Candidates. On the circle, $\nabla h(x) = (-2x_1, -2x_2)^\top \neq 0$, so LICQ holds at every feasible point.

The stationarity system of $L = x_1 + x_2 + \mu(2 - x_1^2 - x_2^2)$ is

$$1 - 2\mu x_1 = 0, \qquad 1 - 2\mu x_2 = 0, \qquad x_1^2 + x_2^2 = 2,$$

with the two solutions $(x^*, \mu^*) = \left((1,1)^\top, \frac{1}{2}\right)$ and $\left((-1,-1)^\top, -\frac{1}{2}\right)$. □

The curvature comes from the constraint

Example 4.7 (Classifying two candidates on a circle)

Consider max $x_1 + x_2$ subject to $h(x) = 2 - x_1^2 - x_2^2 = 0$. Find the candidates, classify them, and certify the global maximizer.

Solution.

Classification. The objective is linear, so $H_f = 0$: all curvature comes from the constraint term, $H = \mu^* H_h = -2\mu^* I$.

- $H = -I$ and $H = I$ at the two candidates, with $T = \{v : v_1 + v_2 = 0\}$ at both.
- On T: $v^\top(-I)v < 0$ for $v \neq 0$, a strict local maximum; $v^\top I v > 0$, a strict local minimum.

$\square$

Certifying the global maximizer

Example 4.7 (Classifying two candidates on a circle)

Consider max $x_1 + x_2$ subject to $h(x) = 2 - x_1^2 - x_2^2 = 0$. Find the candidates, classify them, and certify the global maximizer.

Solution.

Certification. The circle is compact, so Weierstrass applies.

- A global maximizer and a global minimizer exist.
- By the Lagrange multiplier theorem, both are among the two candidates.

Comparing values, $f(1, 1) = 2$ and $f(-1, -1) = -2$: the local classification is also the global one. $\square$

Inequality constraints need a cone

There is also a second-order local test for inequality constraints, but it goes beyond the scope of this course.

- The relevant set of directions is a cone rather than a subspace.

The optimization workflow

Lectures 6 and 7 together give one method for any optimization problem in this course.

1. *Standardize.* Write equalities as $h(x) = 0$ and inequalities as $g_i(x) \geq 0$.
2. *Existence.* Use Weierstrass on a compact feasible set.
    - Use coercivity on an unbounded one, or a direct argument.

The optimization workflow, continued

3. *Candidates.* At interior points of an unconstrained region, solve $\nabla f = 0$.
   - With constraints, guess active sets and check LICQ.
   - Then solve stationarity, feasibility, signs, and complementary slackness.
4. *Certify.* In a concave program, any KKT point is a global maximizer.
   - This is the sufficiency theorem.
   - Otherwise compare candidate values directly.
   - The second-order test gives a local classification instead.
5. *Conclude.* State uniqueness only with strict concavity or a direct argument.
   - Interpret multipliers as marginal values only with the caveats of Lecture 8.
   - Those caveats, from the envelope theorem, are differentiability and a fixed active set.