Linear Algebra

Lecture 3

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Econ 8000 — Math Camp

Conventions

Throughout this lecture, vectors are columns.

For vectors $x, y \in \mathbb{R}^n$, $x \geq y$ means $x_i \geq y_i$ in every coordinate.

Matrix algebra

Matrices and transpose

Definition 1.1 (Matrices and transpose)

An $m \times n$ *matrix* $A = (a_{ij})$ is a rectangular array with m rows and n columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The entry a_{ij} sits in row i and column j.

Two matrices are equal when their dimensions and all corresponding entries agree.

The *transpose* of A is the $n \times m$ matrix $A^\top$ defined by $(A^\top)_{ij} = a_{ji}$.

Square, diagonal, and symmetric matrices

Definition 1.1 (Matrices and transpose)

A matrix is *square* if it has the same number of rows and columns.

It is *diagonal* if every off-diagonal entry is zero, and *symmetric* if $A = A^\top$.

Matrix arithmetic

Matrices of the same dimensions are added entrywise, and so are scalar multiples:

$$(A+B)_{ij} = a_{ij} + b_{ij}, \qquad (\alpha A)_{ij} = \alpha a_{ij}.$$

The $m \times n$ *zero matrix* has every entry zero.

The $n \times n$ *identity matrix* I_n has ones on the diagonal and zeros elsewhere.

The diagonal matrix with diagonal entries $d_1, \ldots, d_n$ is written $\mathrm{diag}(d_1, \ldots, d_n)$.

Adding, scaling, and transposing

- Example: Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 & 1 \\ 2 & -1 & 0 \end{pmatrix}.$$

  Then

$$A + B = \begin{pmatrix} 1 & 3 & 1 \\ 1 & -1 & 3 \end{pmatrix}, \qquad 2A = \begin{pmatrix} 2 & 4 & 0 \\ -2 & 0 & 6 \end{pmatrix}, \qquad A^\top = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Matrix multiplication

Multiplication requires matrix dimensions to be *conformable*.

Definition 1.3 (Matrix multiplication)

If A is $m \times n$ and B is $n \times p$, then their product AB is the $m \times p$ matrix with

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Multiplying two matrices

Each entry of AB is a row of A multiplied against a column of B.

- Example: Let
$$C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$
Then C is 2×3 and D is 3×2, so the product CD is 2×2:
$$CD = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.$$

The reverse product

- Example: Let

$$C = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

  The reverse product is also defined, but it is different in both size and entries:

$$DC = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 1 \end{pmatrix}.$$

Since $CD \neq DC$, matrix multiplication is generally not commutative.

Rules for matrix products

Whenever the products are conformable,

$$(AB)C = A(BC), \qquad A(B + C) = AB + AC, \qquad (AB)^\top = B^\top A^\top,$$

and the identity matrices satisfy $AI_n = A$ and $I_m A = A$ for an $m \times n$ matrix A.

Vectors as matrices

Vectors can be treated as matrices with one column.

- For $x, y \in \mathbb{R}^n$, the product $x^\top y$ is a 1×1 matrix whose entry is $x \cdot y$.
- The product $xy^\top$ is an $n \times n$ matrix.

Linear combinations

Definition 1.5 (Linear combination)

Let $v_1, \ldots, v_k \in \mathbb{R}^n$ be a collection of vectors.

A *linear combination* of $v_1, \ldots, v_k$ is a vector $\sum_{i=1}^k \alpha_i v_i$ with $\alpha_i \in \mathbb{R}$.

Below is a collection of vectors we will use to illustrate concepts in this lecture.

Example 1.6

Let $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Here $v_3 = v_1 + v_2$, so v_3 is a linear combination of v_1 and v_2, with $\alpha_1 = \alpha_2 = 1$.

Subspace and span

Definition 1.7 (Subspace and span)

Let $V \subseteq \mathbb{R}^n$ and let $v_1, \ldots, v_k \in \mathbb{R}^n$.

- The set V is a *subspace* of $\mathbb{R}^n$ if it is nonempty and

$$x, y \in V, \quad \alpha, \beta \in \mathbb{R} \quad \Longrightarrow \quad \alpha x + \beta y \in V.$$

- Their *span* is the set of all such combinations:

$$\mathrm{span}\{v_1, \ldots, v_k\} = \left\{ \sum_{i=1}^{k} \alpha_i v_i : \alpha_1, \ldots, \alpha_k \in \mathbb{R} \right\}.$$

Spans and lines

Every span is a subspace.

- Adding two linear combinations of $v_1, \ldots, v_k$ produces another such combination.
- So does multiplying one of them by a scalar.

The span of one nonzero vector v is the line $\{\alpha v : \alpha \in \mathbb{R}\}$ through the origin.

- A line that does not go through the origin cannot be a subspace.
- Every subspace contains the zero vector.

Independence, basis, and dimension

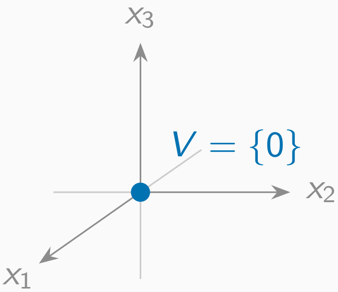
Definition 1.8 (Independence, basis, and dimension)

Let $V \subseteq \mathbb{R}^n$ be a subspace.

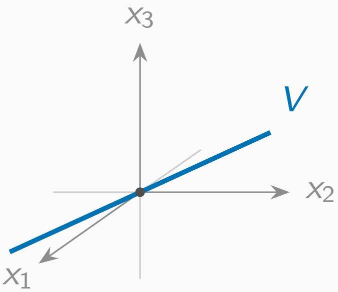
- The collection $v_1, \ldots, v_k$ is *linearly independent* if
$$\alpha_1 v_1 + \cdots + \alpha_k v_k = 0 \quad \Longrightarrow \quad \alpha_1 = \cdots = \alpha_k = 0$$
for all $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, and *linearly dependent* otherwise.
- A *basis* of V is a linearly independent collection of vectors that spans V.
- The *dimension* $\dim V$ of V is the number of vectors in a basis of V.

- One could think of linear dependence as redundancy — if vectors are dependent, then one of them is a linear combination of the others.

All subspaces of $\mathbb{R}^3$, by dimension: the origin and lines

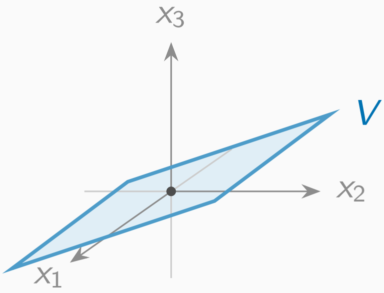


(a) $\dim V = 0$:
the origin

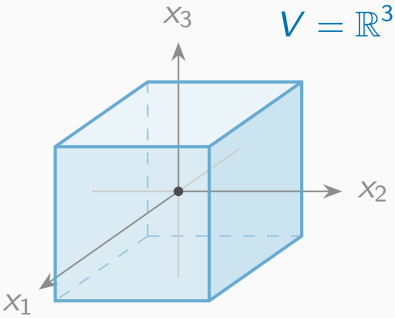


(b) $\dim V = 1$:
a line through 0

All subspaces of $\mathbb{R}^3$, by dimension: planes and all of $\mathbb{R}^3$



(c) $\dim V = 2$:
a plane through 0



(d) $\dim V = 3$:
all of $\mathbb{R}^3$

The vectors v_1, v_2, and v_3

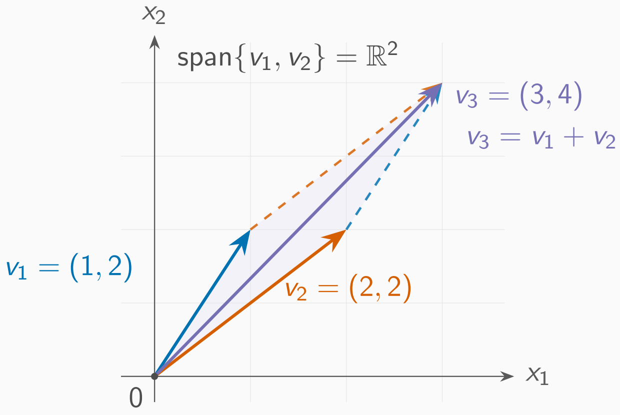


Figure 1.2

- Since $v_3 = v_1 + v_2$, v_3 is redundant and v_1, v_2 form a basis of $\mathbb{R}^2$.

Standard facts and the standard basis

We use the standard finite-dimensional facts about $\mathbb{R}^n$:

- Every subspace of $\mathbb{R}^n$ has a basis.
- Any two bases of the same subspace have the same number of vectors.
- Every independent collection in $\mathbb{R}^n$ has at most n vectors.

The *standard basis* of $\mathbb{R}^n$ is the collection $e_1, \ldots, e_n$.

- Here e_i has 1 in coordinate i and 0 in every other coordinate.

Linear maps and linear systems

From a matrix product to a system

- Example: Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

  Then

$$Ax = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

We call x the *input* and Ax the *output*.

We will also reverse the question: given a target $b \in \mathbb{R}^2$, which inputs x satisfy $Ax = b$?

Rows and columns of A

To understand both the forward calculation $x \mapsto Ax$ and the reverse question $Ax = b$, it helps to name the rows and columns of A.

Write the rows of A as $r_1, r_2, \ldots, r_m$ and its columns as $c_1, c_2, \ldots, c_n$, so that

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{pmatrix} = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}.$$

The column reading of Ax

For $x \in \mathbb{R}^n$, the product Ax can be represented as

$$Ax = \begin{pmatrix} a_{11} \dots a_{1n} \\ \vdots \\ a_{m1} \dots a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} = x_1 c_1 + \cdots + x_n c_n.$$

Thus Ax is a linear combination of the columns of A, with entries of x as weights.

Linear systems

For the row interpretation of matrix A, we need to consider an equation $Ax = b$.

Definition 2.1 (Linear system)

For an $m \times n$ matrix A and $b \in \mathbb{R}^m$, the equation

$$Ax = b, \qquad x \in \mathbb{R}^n,$$

is a *linear system* with *coefficient matrix* A and *right-hand side* b.

It is *consistent* if it has a solution, and *inconsistent* otherwise.

The row reading of $Ax = b$

The row interpretation writes the system as one scalar equation for each row of A:

$$Ax = b \iff r_i x = a_{i1}x_1 + \cdots + a_{in}x_n = b_i, \qquad \text{for each row } i.$$

Each row is thus a restriction on what the input x could be to reach output b.

The column representation asks which linear combination of columns produces b.

Make sure you understand both representations: they are crucial for what comes next.

Example 2.2

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The rows and columns of A are

$$r_1 = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 2 & 2 & 4 \end{pmatrix},$$

and

$$c_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad c_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \qquad c_3 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Two readings of the running example

Example 2.2

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

The column interpretation gives

$$Ax = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

The row interpretation writes $Ax = b$ as

$$x_1 + 2x_2 + 3x_3 = 3, \qquad 2x_1 + 2x_2 + 4x_3 = 4.$$

Linear maps

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear map*, or a *linear transformation*, if

$$T(x+y) = T(x) + T(y), \qquad T(\alpha x) = \alpha T(x)$$

for all $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

Here x is the *input* and $T(x)$ is the *output*.

Theorem 2.4 (Matrices represent linear maps)

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if there is an $m \times n$ matrix A with

$$T(x) = Ax \quad \text{for every } x \in \mathbb{R}^n.$$

In that case A is unique, and its jth column is $T(e_j)$.

If $S : \mathbb{R}^m \to \mathbb{R}^p$ is linear with matrix B, then $S \circ T$ is linear with matrix BA.

Row space and column space

Definition 2.5 (Row and column spaces)

Let A be an $m \times n$ matrix.

Its *row space* and *column space* are

$$\mathrm{Row}(A) = \mathrm{span}\{r_1, \ldots, r_m\} \subseteq \mathbb{R}^n, \qquad \mathrm{Col}(A) = \mathrm{span}\{c_1, \ldots, c_n\} \subseteq \mathbb{R}^m.$$

By the column reading of Ax,

$$\mathrm{Col}(A) = \{Ax : x \in \mathbb{R}^n\},$$

so the column space is the range of the linear map $x \mapsto Ax$.

Rank

The row and column spaces live in different spaces, but have the same dimension.

Theorem 2.6 (Equality of row and column rank)

For every matrix A,

$$\dim \operatorname{Row}(A) = \dim \operatorname{Col}(A).$$

Definition 2.7 (Rank)

The *rank* of A is their common dimension:

$$\operatorname{rank} A = \dim \operatorname{Row}(A) = \dim \operatorname{Col}(A).$$

Changes that preserve the restrictions

Suppose the input x produces the output b, so $Ax = b$.

Let $h \in \mathbb{R}^n$ be a proposed change to the input. The new input is $x' = x + h$.

It produces the same output exactly when

$$A(x') = b \iff A(x+h) = b \iff Ah = 0.$$

Thus h leaves Ax unchanged if and only if $Ah = 0$.

The collection of all such changes is the null space.

Null space

Definition 2.9 (Null space)

For an $m \times n$ matrix A, its *null space*, or *kernel*, is

$$N(A) = \{h \in \mathbb{R}^n : Ah = 0\}.$$

The dimension of $N(A)$ is called the *nullity* of A.

Row interpretation: $Ah = 0 \iff a_{i1}h_1 + \cdots + a_{in}h_n = 0$ for every row i.

- So adding h to x changes the left-hand side of every constraint by zero.

Column interpretation: $Ah = 0 \iff h_1c_1 + \cdots + h_nc_n = 0$.

- Here h_j is the change in the weight on column c_j, and these changes cancel.
- A nonzero such h exists exactly when the columns are linearly dependent.

Rank–nullity theorem

Every null space is a subspace of $\mathbb{R}^n$.

We have now seen two ways to specify a subspace:

- a span uses generating vectors;
- a null space uses equations of the form $Ax = 0$.

Theorem 2.10 (Rank–nullity theorem)

If A is an $m \times n$ matrix, then

$$\operatorname{rank} A + \dim N(A) = n.$$

The rank counts output dimensions; the nullity counts directions that leave Ax unchanged.

The null space of the running example

Example 2.11

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix}$. Calculate $N(A)$.

The null space of the running example

Example 2.11

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \end{pmatrix}$. Calculate $N(A)$.

Solution.

Solving $Ah = 0$ gives

$$h = t \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}, \qquad N(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Thus $\dim N(A) = 1$ and $\operatorname{rank} A + \dim N(A) = 2 + 1 = 3 = n$.

The free scalar t is one degree of freedom.

In general, if $\operatorname{rank} A = r$, there are $n - r$ degrees of freedom. $\square$

The solution set of $Ax = b$: existence

The column interpretation answers the existence question immediately:

$$Ax = b \text{ is consistent} \quad \Longleftrightarrow \quad b \in \mathrm{Col}(A).$$

- By the column reading, $Ax = x_1 c_1 + \cdots + x_n c_n$.
- So $Ax = b$ has a solution exactly when b is a combination of the columns.
- That is, exactly when $b \in \mathrm{span}\{c_1, \ldots, c_n\} = \mathrm{Col}(A)$.

The solution set

The theorem below describes the solution set of $Ax = b$ using the null space.

Theorem 2.12 (Solution set)

If x_0 is one solution of $Ax = b$, then the full solution set is

$$\{x \in \mathbb{R}^n : Ax = b\} = x_0 + N(A), \qquad \text{where } x_0 + N(A) = \{x_0 + h : h \in N(A)\}.$$

Consequently, a consistent system has a unique solution if and only if $N(A) = \{0\}$.

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Consequently, a consistent system has a unique solution if and only if $N(A) = \{0\}$.

Proof.

If $Ax = b = Ax_0$, then $A(x - x_0) = 0$, so $x - x_0 \in N(A)$.

Conversely, if $h \in N(A)$, then $A(x_0 + h) = Ax_0 + Ah = b$. $\square$

The complete description

Together with the column-space criterion, the theorem gives the complete description

$$\{x \in \mathbb{R}^n : Ax = b\} = \begin{cases} \varnothing, & b \notin \mathrm{Col}(A), \\ x_0 + N(A), & b \in \mathrm{Col}(A). \end{cases}$$

In the second case, the number of solutions is 1 if $N(A) = \{0\}$, and infinite otherwise.

Full row and column rank

The preceding results give the two full-rank criteria.

Theorem 2.14 (Full row and column rank)

Let A be $m \times n$.

1. $\operatorname{rank} A = m$ *if and only if* $Ax = b$ *has at least one solution for every* $b \in \mathbb{R}^m$.
2. $\operatorname{rank} A = n$ *if and only if* $Ax = b$ *has at most one solution for every* $b \in \mathbb{R}^m$.

- The first clause follows because $\operatorname{rank} A = m$ exactly when $\operatorname{Col}(A) = \mathbb{R}^m$.
- For the second clause, rank–nullity gives $\operatorname{rank} A = n$ if and only if $N(A) = \{0\}$.
    - The solution-set theorem then says that a consistent system has exactly one solution.

Square matrices and invertibility

Inverse

Definition 3.1 (Inverse)

Let A be an $n \times n$ matrix.

It is *invertible*, or *nonsingular*, if there is an $n \times n$ matrix A^{-1} with

$$A^{-1}A = AA^{-1} = I_n.$$

If no inverse exists, A is *singular*.

An inverse is unique.

- If B and C are both inverses of A, then $B = B(AC) = (BA)C = C$.

Invertible-matrix equivalences

Theorem 3.2 (Invertible-matrix equivalences)

For an $n \times n$ matrix A, the following are equivalent:

1. *A is invertible;*
2. $\operatorname{rank} A = n$*;*
3. *the columns of A are linearly independent;*
4. *$N(A) = \{0\}$;*
5. *for every $b \in \mathbb{R}^n$, the system $Ax = b$ has exactly one solution.*

When these conditions hold, the solution is $x = A^{-1}b$.

- The equivalence of (2)–(5) follows from the rank and solution-set theorems.
- Multiplying $Ax = b$ by A^{-1} gives the unique solution $x = A^{-1}b$.

Determinant

The determinant is a function that turns a square matrix into a scalar.

Definition 3.3 (Determinant)

For a 1×1 matrix, $\det(a) = a$.

For $n \geq 2$, let A_{1j} be the matrix obtained from A by deleting row 1 and column j.

The *determinant* of A is

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}.$$

- This is called *cofactor expansion along the first row*.
- Expanding along any other row or column gives the same number.

Example 3.4 (2×2 determinant)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Here $A_{11} = (d)$ and $A_{12} = (c)$, so

$$\det A = ad - bc.$$

Determinants of small matrices

Example 3.4 (2×2 determinant)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Here $A_{11} = (d)$ and $A_{12} = (c)$, so

$$\det A = ad - bc.$$

Example 3.5 (3×3 determinant)

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Each term now needs a 2×2 determinant:

$$\det A = a(ei - fh) - b(di - fg) + c(dh - eg).$$

Determinant criterion

The determinant supplies a scalar test for the equivalent conditions above.

Theorem 3.6 (Determinant criterion)

A square matrix A is invertible if and only if $\det A \neq 0$.

The inverse of a 2×2 matrix

Example 3.7 (The inverse in the 2×2 case)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find A^{-1} when it exists.

The inverse of a 2×2 matrix

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Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find A^{-1} when it exists.

Solution.

If $\det A \neq 0$, multiplying in either order gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = (ad - bc)I_2.$$

As long as $ad - bc \neq 0$, we can divide by it to get the inverse:

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

$\square$

Eigenvalues and eigenvectors

Eigenvalues and eigenvectors

Definition 4.1 (Eigenvalues and eigenvectors)

Let A be a real $n \times n$ matrix.

A scalar $\lambda \in \mathbb{R}$ is an *eigenvalue* of A if there is a nonzero $v \in \mathbb{R}^n$ with

$$Av = \lambda v.$$

Such a vector v is an *eigenvector* associated with λ.

The eigenvalue equation can be rewritten as

$$Av = \lambda v \quad \iff \quad (A - \lambda I)v = 0.$$

Eigenvalue equivalences

Theorem 4.2 (Eigenvalue equivalences)

For a real $n \times n$ matrix A and $\lambda \in \mathbb{R}$, the following are equivalent:

1. λ *is an eigenvalue of* A;
2. $N(A - \lambda I) \neq \{0\}$;
3. $A - \lambda I$ *is singular;*
4. $\det(A - \lambda I) = 0$.

Corollary 4.3 (Zero eigenvalue)

A square matrix A is singular if and only if 0 is an eigenvalue of A.

Equivalently, A is invertible if and only if all of its eigenvalues are nonzero.

The characteristic equation

The equation

$$\det(A - \lambda I) = 0$$

is the *characteristic equation*.

- Its left-hand side is the *characteristic polynomial*.

The roots of the characteristic polynomial are the eigenvalues of A.

Eigenvalues of a 2×2 matrix

Example 4.4 (Eigenvalues and eigenvectors of a 2×2 matrix)

Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

Eigenvalues of a 2×2 matrix

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Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$.

Solution.

Subtracting λ from each diagonal entry and taking the determinant gives

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{pmatrix}$$
$$= (1-\lambda)(3-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1).$$

Its roots are the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$. □

Eigenvectors of a 2×2 matrix

Example 4.4

Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$, whose eigenvalues are 5 and -1. Find the associated eigenvectors.

Eigenvectors of a 2×2 matrix

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Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$, whose eigenvalues are 5 and -1. Find the associated eigenvectors.

Solution.

For $\lambda = 5$,

$$A - 5I = \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}, \qquad N(A - 5I) = \mathrm{span}\{(1,2)^\top\}.$$

For $\lambda = -1$,

$$A + I = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}, \qquad N(A + I) = \mathrm{span}\{(1,-1)^\top\}.$$

$\square$

Remark 4.5 (How many eigenvalues)

The characteristic polynomial of an $n \times n$ matrix has degree n.

So A has at most n distinct real eigenvalues.

It can have fewer for two separate reasons.

- A root may repeat.
- A root may fail to be real.

Trace

Definition 4.8 (Trace)

The *trace* of an $n \times n$ matrix A is the sum of its diagonal entries,

$$\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}.$$

Theorem 4.9 (Trace and determinant)

Let the characteristic polynomial of A factor into real linear terms,

$$\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda), \qquad \lambda_1, \ldots, \lambda_n \in \mathbb{R}.$$

Then

$$\sum_{i=1}^{n} \lambda_i = \operatorname{tr} A, \qquad \prod_{i=1}^{n} \lambda_i = \det A.$$

Diagonalization

Definition 4.10 (Diagonalizable matrix)

An $n \times n$ A is *diagonalizable* if $A = PDP^{-1}$ for an invertible P and a diagonal D.

Theorem 4.11 (Diagonalization)

A is diagonalizable if and only if it has n independent eigenvectors $v^1, \dots, v^n$.

In that case one may take

$$P = [\, v^1 \ \cdots \ v^n \,], \qquad D = \operatorname{diag}(\lambda_1, \dots, \lambda_n),$$

where λ_i is the eigenvalue belonging to v^i.

In particular, A is diagonalizable whenever it has n distinct real eigenvalues.

Symmetric matrices always have n independent eigenvectors.