Real Analysis in $\mathbb{R}^n$

Lecture 2

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Econ 8000 — Math Camp

Motivation

In economics, we often want to maximize a function subject to constraints:

$$\max_{x \in D} f(x).$$

Here f is the objective function and D is the *feasible set*.

- The feasible set is the set of choices satisfying the constraints.

There is little point in trying to solve this problem if no solution exists.

- This lecture develops the machinery to guarantee that a solution exists, in $\mathbb{R}^n$.

Distance and sequences

Euclidean distance

We begin by defining Euclidean distance in $\mathbb{R}^n$.

- There are other ways to measure and define distance, even in $\mathbb{R}^n$.
- We opt to only use Euclidean distance for most of this class.

Definition 1.1 (Inner product, norm, and distance)

For $x, y \in \mathbb{R}^n$,

- the *inner product* (or *dot product*) of x and y is $x \cdot y = \sum_{i=1}^n x_i y_i$,
- the (Euclidean) *norm* (or *length*) of x is $\|x\| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^n x_i^2}$,
- the *distance* between x and y is $d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

Norm and distance

In $\mathbb{R}^1$ the norm is the absolute value, so $d(x, y) = |x - y|$.

- Example: The vector $x = (3, 4) \in \mathbb{R}^2$ has $x \cdot x = 25$ and $\|x\| = 5$.
- Example: The distance between $(1, 2)$ and $(4, 6)$ is 5 as well.

Cauchy–Schwarz

The inner product and the norm are related by the Cauchy–Schwarz inequality.

Theorem 1.2 (Cauchy–Schwarz)

For every $x, y \in \mathbb{R}^n$,

$$(x \cdot y)^2 \leq (x \cdot x)(y \cdot y),$$

and hence $|x \cdot y| \leq \|x\|\|y\|$.

Properties of the norm

Here are some useful properties of the norm.

Theorem 1.3 (Norm properties)

For $x, y \in \mathbb{R}^n$ and $a \in \mathbb{R}$,

1. $\|x\| \geq 0$, *and* $\|x\| = 0$ *if and only if* $x = 0$ *(positivity),*
2. $\|ax\| = |a|\|x\|$ *(homogeneity),*
3. $\|x + y\| \leq \|x\| + \|y\|$ *(triangle inequality).*

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Proof.

Parts (1) and (2) follow from the definition of the norm, and for part (3),

$$\|x + y\|^2 = \|x\|^2 + 2\, x \cdot y + \|y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2,$$

The first inequality uses Cauchy–Schwarz, and square roots give the result. $\square$

Distance as a metric

The three norm properties say that the distance behaves as a distance should:

$$d(x, y) \geq 0, \quad \text{with equality exactly when } x = y,$$

$$d(x, y) = d(y, x), \qquad d(x, z) \leq d(x, y) + d(y, z).$$

- Nonnegativity is part (1) applied to $x - y$.
- Symmetry is part (2) with $a = -1$.
- The triangle inequality is part (3) applied to $x - z = (x - y) + (y - z)$.

A function satisfying these three conditions is called a *metric*.

Open balls

We can now name the set of points lying within a given distance of a point.

Definition 1.4 (Open ball)

For $x \in \mathbb{R}^n$ and $r > 0$, the *open ball* with center x and radius r is

$$B(x, r) = \{y \in \mathbb{R}^n : \|y - x\| < r\}.$$

- Example: In $\mathbb{R}$ the ball $B(x, r)$ is the interval $(x - r, x + r)$.
- Example: In $\mathbb{R}^2$ it is the disk of radius r around x without its bounding circle.

Interior, open sets, and limit points

Definition 1.5

Let $D \subseteq \mathbb{R}^n$.

- $x \in D$ is an *interior point* if $B(x, r) \subseteq D$ for some $r > 0$. The set of all such points is $\operatorname{int} D$.
- D is *open* if every point of D is interior.
- $a \in \mathbb{R}^n$ is a *limit point* of D if

$$B(a, r) \cap \left(D \setminus \{a\}\right) \neq \emptyset \qquad \text{for every } r > 0.$$

A limit point of D is a point with other points of D arbitrarily close to it.

If $a < b$, then (a, b) is open and

$$\operatorname{int}[a, b] = (a, b).$$

The limit points of (a, b) are exactly the points of $[a, b]$, including both endpoints.

Sequences and convergence

Definition 1.6 (Sequence and convergence)

A *sequence* in $\mathbb{R}^n$ is a list $(x_k)_{k=1}^{\infty}$ with $x_k \in \mathbb{R}^n$.

It *converges* to $x \in \mathbb{R}^n$, written $x_k \to x$, if for every $\varepsilon > 0$ there is $K \in \mathbb{N}$ such that

$$k \geq K \quad \Longrightarrow \quad \|x_k - x\| < \varepsilon.$$

The point x is the *limit* of the sequence.

A sequence that has no limit in $\mathbb{R}^n$ *diverges*.

- Every ball $B(x, \varepsilon)$, however small, contains all terms from some index on.

A convergent sequence

We check the definition on two sequences in $\mathbb{R}$.

Example 1.7

The sequence $x_k = 1/k$ converges to 0.

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Proof.

Let $\varepsilon > 0$ and take an integer $K > 1/\varepsilon$.

Then every $k \geq K$ satisfies

$$|x_k - 0| = \frac{1}{k} \leq \frac{1}{K} < \varepsilon,$$

which is the implication the definition requires of K.

Since $\varepsilon > 0$ was arbitrary, $x_k \to 0$. $\square$

A divergent sequence

Example 1.7

The sequence $x_k = (-1)^k$ diverges.

A divergent sequence

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Proof.

Let $x \in \mathbb{R}$ be a candidate limit and take $\varepsilon = 1$.

By the triangle inequality, $|1 - x| + |-1 - x| \geq |(1 - x) - (-1 - x)| = 2$.

Hence at least one of the two numbers is at least 1.

The values 1 and -1 both occur at arbitrarily large indices.

So every $K \in \mathbb{N}$ admits some $k \geq K$ with $|x_k - x| \geq 1$.

The implication in the definition fails for $\varepsilon = 1$.

Since x was arbitrary, no limit exists. $\square$

Subsequences

A subsequence is exactly what it sounds like.

- It is a sequence drawn from another sequence by skipping some terms.

Definition 1.8 (Subsequence)

Let $k_1 < k_2 < \cdots$ be a strictly increasing sequence of indices.

Then $(x_{k_j})_{j=1}^{\infty}$ is a *subsequence* of (x_k).

Here are some useful properties of sequences and their limits.

Theorem 1.9 (Basic limit facts)

Let (x_k) be a sequence in $\mathbb{R}^n$.

1. *If $x_k \to x$ and $x_k \to y$, then $x = y$.*
2. *If $x_k \to x$, then (x_k) is bounded: there is $M > 0$ with $\|x_k\| \leq M$ for every k.*
3. *The convergence $x_k \to x$ holds if and only if $x_{k,i} \to x_i$ in $\mathbb{R}$ for every $i = 1, \ldots, n$.*
4. *If $x_k \to x$, then every subsequence of (x_k) also converges to x.*

Algebra of limits

Limits also respect the arithmetic of $\mathbb{R}^n$, one operation at a time.

Theorem 1.10 (Algebra of limits)

Suppose $x_k \to x$ and $y_k \to y$ in $\mathbb{R}^n$, and $a_k \to a$ and $b_k \to b$ in $\mathbb{R}$. Then

1. $x_k + y_k \to x + y$,
2. $a_k x_k \to ax$,
3. $a_k b_k \to ab$,
4. $x_k \cdot y_k \to x \cdot y$,
5. $\|x_k\| \to \|x\|$,
6. $a_k / b_k \to a/b$, *provided* $b \neq 0$ *and* $b_k \neq 0$ *for every* k,
7. $x \geq y$, *provided* $x_k \geq y_k$ *coordinatewise for every* k.

Suprema and compactness

Definition 2.1 (Bounds and extrema)

Let $A \subseteq \mathbb{R}$ be nonempty.

- A number u is an *upper bound* of A if $a \leq u$ for every $a \in A$.
- The set A is *bounded above* if it has an upper bound.
- The *supremum*, or *least upper bound*, of A is written $\sup A$.
- It is an upper bound s of A satisfying $s \leq u$ for every upper bound u of A.
- The *maximum* of A, written $\max A$, is an upper bound z of A with $z \in A$.

Lower bounds and bounded below reverse every inequality above.

So do the *infimum* $\inf A$ and the *minimum* $\min A$.

Maxima and suprema

If a maximum exists, it is automatically the supremum.

Conversely, if the supremum exists and belongs to A, it is the maximum.

Thus

$$\max A \text{ exists} \quad \Longleftrightarrow \quad \sup A \text{ exists and } \sup A \in A,$$

and then the two numbers are equal.

Suprema of intervals

- Example: The upper bounds of $(0, 1)$ are precisely the numbers at least 1.
  - So $\sup(0, 1) = 1$ and $\inf(0, 1) = 0$.
  - Neither bound belongs to the interval, so it has neither a maximum nor a minimum.
- Example: The interval $[0, 1]$ has the same supremum and infimum.
  - It contains both, so $\max[0, 1] = 1$ and $\min[0, 1] = 0$.

Completeness of $\mathbb{R}$

This definition does not guarantee that a supremum exists.

- It is possible for a set to have upper bounds but no least one.
- Example: The set of rational q with $q^2 < 2$ is bounded above by 2.
    - It has no least upper bound in $\mathbb{Q}$.

When dealing with $\mathbb{R}$, we make a fundamental assumption.

Axiom 2.2 (Completeness of $\mathbb{R}$)

Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

Closed, bounded, and compact sets

Definition 2.3 (Closed, bounded, and compact sets)

A set $D \subseteq \mathbb{R}^n$ is

- *closed* if $x_k \in D$ and $x_k \to x$ imply $x \in D$;
- *bounded* if there is $M > 0$ such that $\|x\| \leq M$ for every $x \in D$;
- *compact* if every sequence in D has a subsequence converging to a point of D.

- Compactness lets a sequence in D produce a limit still in D.

Theorem 2.4 (Heine–Borel)

A set $D \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.

- Thus every closed interval $[a, b]$ is compact, whereas $(0, 1)$ and $\mathbb{R}$ are not.
- The empty set is compact, so compactness alone does not supply a feasible point.

Continuity and existence

Limits of functions

A function limit describes what happens to outputs as the input approaches a point.

Definition 3.1 (Limit of a function)

Let $D \subseteq \mathbb{R}^n$, let $f : D \to \mathbb{R}^m$, let a be a limit point of D, and let $L \in \mathbb{R}^m$. We write

$$\lim_{x \to a} f(x) = L$$

if every sequence (x_k) in $D \setminus \{a\}$ with $x_k \to a$ satisfies $f(x_k) \to L$.

- The point a need not be in the domain and value $f(a)$ is not relevant.

One-sided limits

For a function of one real variable,

$$\lim_{x \downarrow a} f(x) = L \qquad \text{and} \qquad \lim_{x \uparrow a} f(x) = L$$

denote the *right-hand limit* and *left-hand limit*, respectively.

- $x \downarrow a$: restrict to sequences with $x_k > a$.
- $x \uparrow a$: restrict to sequences with $x_k < a$.
- If the two-sided limit exists, every one-sided limit that is defined exists and has the same value.

Continuity

Definition 3.2 (Continuity)

Let $D \subseteq \mathbb{R}^n$, let $f : D \to \mathbb{R}^m$, and let $x \in D$. The function f is *continuous at* x if, for every sequence (x_k) in D,

$$x_k \to x \quad \Longrightarrow \quad f(x_k) \to f(x).$$

It is *continuous on* D if this holds at every $x \in D$.

If x is also a limit point of D, continuity at x is equivalent to

$$\lim_{y \to x} f(y) = f(x).$$

Equivalently, for every $\varepsilon > 0$ there is $\delta > 0$ such that, for every $y \in D$,

$$\|y - x\| < \delta \quad \Longrightarrow \quad \|f(y) - f(x)\| < \varepsilon.$$

Operations preserving continuity

Here are the rules that let us build continuous functions out of simpler ones.

Theorem 3.3 (Operations preserving continuity)

Let $D \subseteq \mathbb{R}^n$.

1. *If $f, g : D \to \mathbb{R}$ are continuous and $a \in \mathbb{R}$, then $f + g$, af, and fg are continuous.*
    - *The quotient f/g is continuous on the subset of D on which $g \neq 0$.*
2. *A composition of continuous functions is continuous.*

- Constants and $f(x) = x$ are continuous, hence so is every polynomial.
- With exp and log continuous, e^{-x^2} and $\log(1 + x^2)$ etc are continuous.

Definition 3.4 (Maximizers and minimizers)

Let $D \subseteq \mathbb{R}^n$ be nonempty and let $f : D \to \mathbb{R}$.

The set of maximizers of f on D is

$$\arg\max_{x \in D} f(x) = \{x^* \in D : f(x^*) \geq f(x) \text{ for every } x \in D\}.$$

The set $\arg\min_{x \in D} f(x)$ of minimizers reverses the inequality.

Weierstrass

The main existence result is the Weierstrass theorem.

Theorem 3.5 (Weierstrass)

Let $D \subseteq \mathbb{R}^n$ be nonempty and compact, and let $f : D \to \mathbb{R}$ be continuous.

Then f attains a maximum and a minimum on D; equivalently,

$$\operatorname*{arg\,max}_{x \in D} f(x) \neq \emptyset, \qquad \operatorname*{arg\,min}_{x \in D} f(x) \neq \emptyset.$$

- The three hypotheses are nonemptiness, compactness, and continuity.
- Failure of any one of them can leave f without a maximum or minimum.
- A maximum might still exist when one fails, but then it needs another proof.

One failed hypothesis at a time

None of the following problems has a maximizer.

- In each one exactly one hypothesis of the Weierstrass theorem fails.

Example 3.6

In $\mathbb{R}^2$, let $f(x) = x_1 + x_2$ on

$$D = \{x \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1,\ x_1 x_2 \geq 1\}.$$

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Proof.

The feasible set D is empty, so f has no maximizer. $\square$

A domain that is not closed

Example 3.6

Let $D = (0, 1)$ and $f(x) = x$.

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Proof.

The domain is bounded but not closed, so again it is not compact.

The sequence $x_k = 1/(k+1)$ lies in D and converges to $0 \notin D$.

Here $\sup f(D) = 1$, while $f(x) < 1$ for every $x \in D$. $\square$

A function that is not continuous

Example 3.6

Let $D = [0, 1]$, which is compact, and

$$f(x) = \begin{cases} x, & 0 \leq x < 1, \\ 0, & x = 1. \end{cases}$$

A function that is not continuous

Example 3.6

Let $D = [0, 1]$, which is compact, and

$$f(x) = \begin{cases} x, & 0 \le x < 1, \\ 0, & x = 1. \end{cases}$$

Proof.

What fails is continuity, and only at 1.

The sequence $x_k = 1 - 1/k$ lies in D and converges to 1.

But $f(x_k) = 1 - 1/k \to 1 \neq 0 = f(1)$, which the definition forbids.

Again $\sup f(D) = 1$ is not attained. $\square$

Intermediate value theorem

We can already prove useful results about continuous functions in $\mathbb{R}$.

- A continuous function takes every value between its endpoint values.

Theorem 3.7 (Intermediate value theorem)

Let $f : [a, b] \to \mathbb{R}$ be continuous. If

$$\min\{f(a), f(b)\} \leq c \leq \max\{f(a), f(b)\},$$

then there is $x \in [a, b]$ such that $f(x) = c$.

Brouwer in $\mathbb{R}$

From that, we obtain our first *fixed point theorem* in $\mathbb{R}$ for free.

Theorem 3.8 (Brouwer in $\mathbb{R}$)

Let $a \leq b$ and let $f : [a, b] \to [a, b]$ be continuous.

Then f has a fixed point: there is $x^* \in [a, b]$ such that $f(x^*) = x^*$.

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Then f has a fixed point: there is $x^* \in [a, b]$ such that $f(x^*) = x^*$.

Proof.

Let $g(x) = f(x) - x$, which is continuous on $[a, b]$ by part (1) above.

Since f takes values in $[a, b]$, we have $g(a) = f(a) - a \geq 0$ and $g(b) = f(b) - b \leq 0$.

So $g(b) \leq 0 \leq g(a)$, and the intermediate value theorem applies to g with $c = 0$.

It gives $x^* \in [a, b]$ with $g(x^*) = 0$, which means $f(x^*) = x^*$. $\square$