Lecture 2: Real Analysis in $\mathbb{R}^n$

In economics, we often want to maximize a function subject to constraints:

$$\max_{x \in D} f(x).$$

Here f is the objective function and D is the *feasible set*: the set of choices satisfying the constraints. There is little point in trying to solve this problem if no solution exists. This lecture develops the machinery to guarantee that a solution exists, in $\mathbb{R}^n$.

1 Distance and sequences

We begin by defining Euclidean distance in $\mathbb{R}^n$. Note that there are other ways to measure and define distance, even in $\mathbb{R}^n$. We opt to only use Euclidean distance for most of this class.

Definition 1.1 (Inner product, norm, and distance)**.** For $x, y \in \mathbb{R}^n$,

- the *inner product* (also known as the *dot product*) of x and y is

$$x \cdot y = \sum_{i=1}^{n} x_i y_i,$$

- the (Euclidean) *norm* (or *length*) of x is

$$\|x\| = \sqrt{x \cdot x} = \sqrt{\sum_{i=1}^{n} x_i^2},$$

- the *distance* between x and y is

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

For example, $x = (3, 4) \in \mathbb{R}^2$ has $x \cdot x = 25$ and $\|x\| = 5$, and the distance between $(1, 2)$ and $(4, 6)$ is 5 as well. In $\mathbb{R}^1$ the norm is the absolute value, so $d(x, y) = |x - y|$.

The inner product and the norm are related by the Cauchy–Schwarz inequality:

Theorem 1.2 (Cauchy–Schwarz)**.** *For every $x, y \in \mathbb{R}^n$,*

$$(x \cdot y)^2 \leq (x \cdot x)(y \cdot y),$$

and hence $|x \cdot y| \leq \|x\| \|y\|$.

Proof (optional). If $y = 0$, both sides of the first inequality vanish.

Next suppose $y \neq 0$, so that $y \cdot y = \sum_{i=1}^n y_i^2 > 0$. For every $\lambda \in \mathbb{R}$, consider a vector $z = x - \lambda y$. By Definition 1.1, $z \cdot z = \sum_{i=1}^n z_i^2 \geq 0$. Hence, we have

$$0 \leq z \cdot z = (x - \lambda y) \cdot (x - \lambda y) = x \cdot x - 2\lambda\,(x \cdot y) + \lambda^2\,(y \cdot y).$$

Next, choose $\lambda = (x \cdot y)/(y \cdot y)$ so that

$$0 \leq x \cdot x - 2\frac{(x \cdot y)}{(y \cdot y)}\,(x \cdot y) + \left(\frac{(x \cdot y)}{(y \cdot y)}\right)^2 (y \cdot y) = x \cdot x - \frac{(x \cdot y)^2}{y \cdot y}.$$

Multiplying it by $y \cdot y > 0$ gives the first inequality in Theorem 1.2. Taking square roots gives the second inequality in Theorem 1.2, since $(x \cdot x)^{1/2} = \|x\|$ by Definition 1.1. $\square$

Here are some useful properties of the norm.

Theorem 1.3 (Norm properties)**.** *For $x, y \in \mathbb{R}^n$ and $a \in \mathbb{R}$,*

1. $\|x\| \geq 0$*, and* $\|x\| = 0$ *if and only if* $x = 0$ *(positivity),*
2. $\|ax\| = |a|\|x\|$ *(homogeneity),*
3. $\|x + y\| \leq \|x\| + \|y\|$ *(triangle inequality).*

Proof. Parts (1) and (2) follow from Definition 1.1. The triangle inequality is shown as follows:

$$\|x + y\|^2 = \|x\|^2 + 2\,x \cdot y + \|y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2,$$

where the first inequality uses the Cauchy–Schwarz inequality. Taking square roots gives the triangle inequality. $\square$

Read in terms of the distance function d, the three norm properties say that the distance behaves as a distance should: for all $x, y, z \in \mathbb{R}^n$,

$$\begin{aligned} &d(x, y) \geq 0, \quad \text{with equality exactly when } x = y, \\ &d(x, y) = d(y, x), \qquad d(x, z) \leq d(x, y) + d(y, z). \end{aligned}$$

Here, nonnegativity is part (1) applied to $x - y$; symmetry is part (2) with $a = -1$; and the triangle inequality is part (3) applied to $x - z = (x - y) + (y - z)$. A function satisfying these three conditions is called a *metric.*

Equipped with a concept of distance, we can name the set of points lying within a given distance of a point.

Definition 1.4 (Open ball)**.** For $x \in \mathbb{R}^n$ and $r > 0$, the *open ball* with center x and radius r is

$$B(x, r) = \{y \in \mathbb{R}^n : \|y - x\| < r\}.$$

Definition 1.5 (Interior points, open sets, and limit points)**.** Let $D \subseteq \mathbb{R}^n$.

- A point $x \in D$ is an *interior point* of D if $B(x, r) \subseteq D$ for some $r > 0$. The set of all interior points of D is its *interior*, written $\operatorname{int} D$.
- The set D is *open* if every point of D is an interior point.

- A point $a \in \mathbb{R}^n$ is a *limit point* of D if every ball $B(a,r)$ contains a point of D different from a; equivalently,

$$B(a,r) \cap \big(D \setminus \{a\}\big) \neq \emptyset \qquad \text{for every } r > 0.$$

For example, let $a < b$ and take $x \in (a,b)$. The number $r = \min\{x-a, b-x\}$ is positive and $B(x,r) \subseteq (a,b)$, so (a,b) is open. The endpoints of $[a,b]$ are not interior points, and therefore

$$\operatorname{int}[a,b] = (a,b).$$

The limit-point condition says that points of D different from a can be found arbitrarily close to a; the point a need not itself belong to D.

Geometrically, in $\mathbb{R}$ the ball $B(x,r)$ is the interval $(x-r, x+r)$, and in $\mathbb{R}^2$ it is the disk of radius r around x without its bounding circle.

Definition 1.6 (Sequence and convergence)**.** A *sequence* in $\mathbb{R}^n$ is a list $(x_k)_{k=1}^{\infty}$ with $x_k \in \mathbb{R}^n$. It *converges* to $x \in \mathbb{R}^n$, and x is its *limit*, written $x_k \to x$, if for every $\varepsilon > 0$ there is $K \in \mathbb{N}$ such that

$$k \geq K \quad \Longrightarrow \quad \|x_k - x\| < \varepsilon.$$

A sequence that has no limit in $\mathbb{R}^n$ *diverges.*

In the notation of Definition 1.4, the condition says that every ball $B(x,\varepsilon)$, however small its radius, contains all terms of the sequence from some index on.

We check the definition on two sequences in $\mathbb{R}$.

Example 1.7 (Convergence and divergence)**.**

1. The sequence $x_k = 1/k$ converges to 0. Let $\varepsilon > 0$ and take an integer $K > 1/\varepsilon$. Then every $k \geq K$ satisfies

$$|x_k - 0| = \frac{1}{k} \leq \frac{1}{K} < \varepsilon,$$

which is the implication Definition 1.6 requires of K. Since $\varepsilon > 0$ was arbitrary, $x_k \to 0$.

2. The sequence $x_k = (-1)^k$ diverges. Let $x \in \mathbb{R}$ be a candidate limit and take $\varepsilon = 1$. By part (3) of Theorem 1.3,

$$|1 - x| + |-1 - x| \geq |(1-x) - (-1-x)| = 2,$$

so at least one of the two numbers is at least 1. The values 1 and -1 both occur at arbitrarily large indices, so every $K \in \mathbb{N}$ admits some $k \geq K$ with $|x_k - x| \geq 1$, and the implication in Definition 1.6 fails for $\varepsilon = 1$. Since x was arbitrary, no limit exists.

A subsequence is exactly what is sounds like: a sequence drawn from another sequence by skipping some terms.

Definition 1.8 (Subsequence)**.** If $k_1 < k_2 < \cdots$ is a strictly increasing sequence of indices, then $(x_{k_j})_{j=1}^{\infty}$ is a *subsequence* of (x_k).

Here are some useful properties of sequences and their limits.

Theorem 1.9 (Basic limit facts)**.** *Let (x_k) be a sequence in $\mathbb{R}^n$.*

1. *If* $x_k \to x$ *and* $x_k \to y$*, then* $x = y$*.*

2. *If* $x_k \to x$*, then* (x_k) *is bounded: there is* $M > 0$ *with* $\|x_k\| \leq M$ *for every* k*.*

3. *The convergence* $x_k \to x$ *holds if and only if* $x_{k,i} \to x_i$ *in* $\mathbb{R}$ *for every coordinate* $i = 1, \ldots, n$*.*

4. *If* $x_k \to x$*, then every subsequence of* (x_k) *also converges to* x*.*

Proof (optional). Uniqueness. Let $\varepsilon > 0$. By Definition 1.6 there is K_1 with $\|x_k - x\| < \varepsilon/2$ for $k \geq K_1$, and K_2 with $\|x_k - y\| < \varepsilon/2$ for $k \geq K_2$. Fix one $k \geq \max\{K_1, K_2\}$ and write $x - y = (x - x_k) + (x_k - y)$; parts (2) and (3) of Theorem 1.3 give

$$\|x - y\| \leq \|x_k - x\| + \|x_k - y\| < \varepsilon.$$

So the nonnegative number $\|x - y\|$ lies below every positive ε, hence equals 0, and part (1) of Theorem 1.3 gives $x = y$.

Boundedness. Left as an exercise: Definition 1.6 with $\varepsilon = 1$ leaves all but finitely many terms in $B(x, 1)$, and a bound is the largest of $\|x\| + 1$ and the norms of those remaining terms.

Coordinates. Write $a_j = x_{k,j} - x_j$. Since a_i^2 is one of the squares summed in Definition 1.1, and since that sum is at most $\left(\sum_{j=1}^{n} |a_j|\right)^2$, taking square roots gives

$$|x_{k,i} - x_i| \leq \|x_k - x\| \leq \sum_{j=1}^{n} |x_{k,j} - x_j|.$$

Let $\varepsilon > 0$. If $x_k \to x$, the left inequality lets any K that Definition 1.6 supplies for ε serve the coordinate sequence $(x_{k,i})$ as well. Conversely, if every coordinate converges, choose for each j an index K_j beyond which $|x_{k,j} - x_j| < \varepsilon/n$; then $k \geq \max\{K_1, \ldots, K_n\}$ makes the right-hand sum, and hence $\|x_k - x\|$, smaller than ε.

Subsequences. The indices of a subsequence satisfy $k_j \geq j$: this holds at $j = 1$ because indices are natural numbers, and $k_{j+1} > k_j \geq j$ forces $k_{j+1} \geq j + 1$. Now let $\varepsilon > 0$ and take the K that Definition 1.6 supplies for (x_k). Every $j \geq K$ has $k_j \geq j \geq K$, and therefore $\|x_{k_j} - x\| < \varepsilon$. $\square$

Limits also respect the arithmetic of $\mathbb{R}^n$, one operation at a time.

Theorem 1.10 (Algebra of limits)**.** *Suppose* $x_k \to x$ *and* $y_k \to y$ *in* $\mathbb{R}^n$*, and* $a_k \to a$ *and* $b_k \to b$ *in* $\mathbb{R}$*. Then*

1. $x_k + y_k \to x + y$*,*

2. $a_k x_k \to a x$*,*

3. $a_k b_k \to ab$*,*

4. $x_k \cdot y_k \to x \cdot y$*,*

5. $\|x_k\| \to \|x\|$*,*

6. $a_k / b_k \to a/b$*, provided* $b \neq 0$ *and* $b_k \neq 0$ *for every* k*,*

7. $x \geq y$*, provided* $x_k \geq y_k$ *coordinatewise for every* k*.*

Its proof is left as an exercise; the claims follow from part (3) of Theorem 1.3, which gives $|\|x_k\| - \|x\|| \leq \|x_k - x\|$, from Theorem 1.2, and from the coordinatewise criterion in part (3) of Theorem 1.9, together with the boundedness supplied by part (2).

2 Suprema and compactness

We first develop two ingredients for existence arguments: a way to describe the limiting upper value of a set, and a condition that makes sequences behave well inside a feasible set.

2.1 Suprema and maximizing sequences

Definition 2.1 (Bounds and extrema)**.** Let $A \subseteq \mathbb{R}$ be nonempty.

- A number u is an *upper bound* of A if
$$a \le u \qquad \text{for every } a \in A,$$
and A is *bounded above* if it has an upper bound.
- The *supremum*, or *least upper bound*, of A, written $\sup A$, is an upper bound s of A satisfying
$$s \le u \qquad \text{for every upper bound } u \text{ of } A.$$
- A number z is the *maximum* of A, written $\max A$, if
$$z \in A \qquad \text{and} \qquad a \le z \;\; \text{for every } a \in A.$$

Lower bounds, *bounded below*, the *infimum*, or *greatest lower bound*, $\inf A$, and the *minimum* $\min A$ are defined by reversing every inequality above.

If a maximum exists, it is automatically the supremum. Conversely, if the supremum exists and belongs to A, it is the maximum. Thus

$$\max A \text{ exists} \quad \Longleftrightarrow \quad \sup A \text{ exists and } \sup A \in A,$$

and then the two numbers are equal.

For example, the upper bounds of $(0,1)$ are precisely the numbers at least 1, so

$$\sup(0,1) = 1, \qquad \inf(0,1) = 0.$$

Neither bound belongs to the interval, so it has neither a maximum nor a minimum. The interval $[0,1]$ has the same supremum and infimum, but it contains both; hence $\max[0,1] = 1$ and $\min[0,1] = 0$.

Definition 2.1 does not guarantee that a supremum exists; it is possible for a set to have upper bounds but no least one. For instance, the set of rational numbers q with $q^2 < 2$ is bounded above by 2, but it has no least upper bound in $\mathbb{Q}$. When dealing with $\mathbb{R}$, we make a fundamental assumption that every nonempty set that is bounded above has a supremum. This is known as the completeness property of the real numbers.

Axiom 2.2 (Completeness of $\mathbb{R}$)**.** Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

Note the role of both hypotheses of the axiom. First, the set $\mathbb{R}$ is nonempty but not bounded above, so it has no supremum. Second, the empty set $\emptyset$ is bounded above by every real number, but it has no least upper bound.

2.2 Closed, bounded, and compact sets

Our existence argument will turn a sequence of feasible points into a candidate solution. For this to work, some subsequence must converge and its limit must remain in the feasible set. Compactness guarantees both.

Definition 2.3 (Closed, bounded, and compact sets)**.** A set $D \subseteq \mathbb{R}^n$ is

- *closed* if $x_k \in D$ and $x_k \to x$ imply $x \in D$;
- *bounded* if there is $M > 0$ such that $\|x\| \le M$ for every $x \in D$;
- *compact* if every sequence (x_k) in D has a subsequence converging to some point of D.

There are two failures to remember: $x_k = k$ runs off to infinity in $\mathbb{R}$, while $x_k = 1/k$ converges to a point missing from $(0, 1)$. Compactness rules out both.

In Euclidean space these conditions are related by a fundamental result that we use without proof.

Theorem 2.4 (Heine–Borel)**.** *A set $D \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Thus every closed interval $[a, b]$ is compact, whereas $(0, 1)$ and $\mathbb{R}$ are not. The empty set is compact, so compactness alone does not supply a feasible point.

3 Continuity and existence

3.1 Continuity

So far, a limit has described the behavior of a sequence. We will also need to describe the behavior of a function as its input approaches a point.

Definition 3.1 (Limit of a function)**.** Let $D \subseteq \mathbb{R}^n$, let $f : D \to \mathbb{R}^m$, let a be a limit point of D, and let $L \in \mathbb{R}^m$. We write

$$\lim_{x \to a} f(x) = L$$

if, for every sequence (x_k) in $D \setminus \{a\}$ such that $x_k \to a$, we have $f(x_k) \to L$.

In Euclidean space this sequential definition is equivalent to the familiar ε–δ condition: for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$x \in D, \quad 0 < \|x - a\| < \delta \quad \Longrightarrow \quad \|f(x) - L\| < \varepsilon.$$

The restriction $x \neq a$ is deliberate: a function limit depends on values near a, not on the value at a. The point a need not be in the domain. For example, if $q(h) = h^2/|h| = |h|$ for $h \neq 0$, then Theorem 1.10 gives $\lim_{h \to 0} q(h) = 0$, even though $q(0)$ is not defined.

For functions of one real variable, suppose a is approached by points of D from the indicated side. The notation

$$\lim_{x \downarrow a} f(x) = L \qquad \text{and} \qquad \lim_{x \uparrow a} f(x) = L$$

denotes the *right-hand limit* and *left-hand limit*, respectively. The first applies Definition 3.1 to $D \cap (a, \infty)$; the second applies it to $D \cap (-\infty, a)$. If the two-sided limit exists, then every one-sided limit that is defined exists and equals it.

Definition 3.2 (Continuity)**.** Let $D \subseteq \mathbb{R}^n$, let $f : D \to \mathbb{R}^m$, and let $x \in D$. The function f is *continuous at* x if, whenever $x_k \in D$ and $x_k \to x$, we have

$$f(x_k) \to f(x).$$

The function is *continuous on* D, or simply *continuous*, if it is continuous at every point of D.

At a limit point $x \in D$, Definition 3.2 is equivalent to

$$\lim_{y \to x} f(y) = f(x).$$

Equivalently, for every $\varepsilon > 0$ there is $\delta > 0$ such that, for every $y \in D$,

$$\|y - x\| < \delta \quad \Longrightarrow \quad \|f(y) - f(x)\| < \varepsilon.$$

We will continue to use the sequential formulation because it fits our existence argument. In Lecture 4, the same function-limit notation will define derivatives: a difference quotient is evaluated only for $h \neq 0$, while h approaches 0.

Here are the rules that let us build continuous functions out of simpler ones.

Theorem 3.3 (Operations preserving continuity)**.** *Let* $D \subseteq \mathbb{R}^n$*.*

1. *If* $f, g : D \to \mathbb{R}$ *are continuous and* $a \in \mathbb{R}$*, then* $f + g$*,* af*, and* fg *are continuous on* D*, and the quotient* f/g *is continuous on the subset of* D *on which* $g \neq 0$*.*
2. *A composition of continuous functions is continuous.*

Its proof is left as an exercise. Part (1) follows from Definition 3.2 applied to an arbitrary sequence $x_k \to x$, together with Theorem 1.10; part (2) follows from Definition 3.2 applied twice.

For example, constant functions and the identity function $f(x) = x$ are continuous on $\mathbb{R}$ by Definition 3.2. Part (1) then makes every polynomial continuous on $\mathbb{R}$, and every rational function continuous off the zeros of its denominator, such as $(x^2+1)/(x-2)$ on $\mathbb{R} \setminus \{2\}$. Taking as known that exp is continuous on $\mathbb{R}$ and log on $(0, \infty)$, part (2) gives combinations: e^{-x^2} is continuous on $\mathbb{R}$, and so is $\log(1 + x^2)$, since $1 + x^2$ stays positive.

3.2 The Weierstrass theorem

We are now ready to show existence. First, let us formalize what a solution to an optimization problem is.

Definition 3.4 (Maximizers and minimizers)**.** Let $D \subseteq \mathbb{R}^n$ be nonempty and let $f : D \to \mathbb{R}$. The set of maximizers of f on D is

$$\operatorname*{arg\,max}_{x \in D} f(x) = \{x^* \in D : f(x^*) \geq f(x) \text{ for every } x \in D\}.$$

The set $\operatorname{arg\,min}_{x \in D} f(x)$ of minimizers is defined by reversing the inequality.

The main existence result is the Weierstrass theorem, which we can now state and prove.

Theorem 3.5 (Weierstrass)**.** *Let* $D \subseteq \mathbb{R}^n$ *be nonempty and compact, and let* $f : D \to \mathbb{R}$ *be continuous. Then* f *attains a maximum and a minimum on* D*; equivalently,*

$$\operatorname*{arg\,max}_{x \in D} f(x) \neq \emptyset, \qquad \operatorname*{arg\,min}_{x \in D} f(x) \neq \emptyset.$$

Proof (optional). We first prove that f has a maximizer; the minimizer will follow at the end.

Step 1: show that the attainable values $f(D)$ are bounded above. Suppose they were not. Then for each k there would be $y_k \in D$ with $f(y_k) > k$. By Definition 2.3, some subsequence satisfies $y_{k_j} \to y \in D$, and Definition 3.2 gives $f(y_{k_j}) \to f(y)$. Thus $(f(y_{k_j}))$ is bounded by part (2) of Theorem 1.9. But $k_j \geq j$, so $f(y_{k_j}) > k_j \geq j$ for every j, making the same sequence unbounded. This contradiction proves that $f(D)$ is bounded above.

Step 2: construct feasible points whose values approach the best possible value. The set $f(D)$ is nonempty because D is nonempty. By Step 1 it is also bounded above, so completeness gives the real number $S = \sup f(D)$. For each k, the smaller number $S - 1/k$ cannot be an upper bound of $f(D)$. Hence there is $x_k \in D$ with

$$S - \frac{1}{k} < f(x_k) \leq S, \qquad \text{that is,} \qquad |f(x_k) - S| < \frac{1}{k}.$$

Since $1/k \to 0$ by part (1) of Example 1.7, this proves $f(x_k) \to S$. The sequence (x_k) is called a *maximizing sequence.*

Step 3: obtain a feasible candidate for the maximizer. Compactness of D gives a subsequence x_{k_j} and a point $x^* \in D$ such that $x_{k_j} \to x^*$. The membership $x^* \in D$ is important: the candidate remains feasible.

Step 4: prove that the candidate is a maximizer. Continuity gives $f(x_{k_j}) \to f(x^*)$. On the other hand, because $f(x_k) \to S$, its subsequence also satisfies $f(x_{k_j}) \to S$ by part (4) of Theorem 1.9. Limits are unique, so $f(x^*) = S$. Since S is an upper bound of $f(D)$, we have $f(x) \leq S = f(x^*)$ for every $x \in D$. Therefore x^* is a maximizer in the sense of Definition 3.4.

Step 5: obtain a minimizer. The function $-f$ is continuous by part (1) of Theorem 3.3. Applying Steps 1–4 to $-f$ gives a point that maximizes $-f$, which is exactly a point that minimizes f. □

Note that failure of any one of the three hypotheses — nonemptiness, compactness, or continuity — can lead to a function that does not attain its maximum or minimum.

Example 3.6 (One failed hypothesis at a time)**.** None of the following problems has a maximizer, and in each one exactly one hypothesis of Theorem 3.5 fails.

1. In $\mathbb{R}^2$, let $f(x) = x_1 + x_2$ on

   $$D = \{x \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1,\ x_1 x_2 \geq 1\}.$$

   The feasible set D is empty, so f has no maximizer.

2. $D = (0, 1)$ and $f(x) = x$. The domain is bounded but not closed, since $x_k = 1/(k+1)$ lies in D and converges to $0 \notin D$, so again it is not compact. Here $\sup f(D) = 1$, while $f(x) < 1$ for every $x \in D$.

3. $D = [0, 1]$, which is compact, and

   $$f(x) = \begin{cases} x, & 0 \leq x < 1, \\ 0, & x = 1. \end{cases}$$

   What fails is continuity, and only at 1: the sequence $x_k = 1 - 1/k$ lies in D and converges to 1, while $f(x_k) = 1 - 1/k \to 1 \neq 0 = f(1)$, which Definition 3.2 forbids. Again $\sup f(D) = 1$ is not attained.

That said, a maximum might exist even if one of the hypotheses of Theorem 3.5 fails. In that case, one must prove existence by other means, such as by finding a candidate and verifying that it is indeed a maximizer.

3.3 Intermediate values and fixed points

While we need linear algebra to meaningfully study functions in $\mathbb{R}^n$, we can already prove some useful results about continuous functions in $\mathbb{R}$. The intermediate value theorem says that a continuous function on an interval takes on every value between its values at the endpoints.

Theorem 3.7 (Intermediate value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. If*

$$\min\{f(a), f(b)\} \le c \le \max\{f(a), f(b)\},$$

then there is $x \in [a, b]$ such that $f(x) = c$.

Proof (optional). Assume first that $f(a) \le c \le f(b)$, and let

$$A = \{x \in [a, b] : f(x) \le c\}.$$

Then A contains a and is bounded above by b, so $s = \sup A$ exists by Axiom 2.2 and satisfies $a \le s \le b$. We show that $f(s) = c$.

$f(s) \le c$. For each k, the number $s - 1/k$ is smaller than s and therefore not an upper bound of A, so we may pick $a_k \in A$ with

$$s - \frac{1}{k} < a_k \le s.$$

Then $a_k \to s$, since $1/k \to 0$ by part (1) of Example 1.7, and Definition 3.2 gives $f(a_k) \to f(s)$. Each a_k lies in A, so $f(a_k) \le c$ for every k, and part (7) of Theorem 1.10 carries the inequality to the limit: $f(s) \le c$.

$f(s) \ge c$. If $s = b$, this is the hypothesis $f(b) \ge c$. If $s < b$, no point of $(s, b]$ lies in A, so $f(x) > c$ on $(s, b]$. The points $x_k = s + 1/k$ lie in $(s, b]$ as soon as $1/k \le b - s$, and $x_k \to s$, so Definition 3.2 gives $f(x_k) \to f(s)$. Part (7) of Theorem 1.10 again carries $f(x_k) > c$ to the limit, giving $f(s) \ge c$.

This proves $f(s) = c$. In the remaining case $f(b) \le c \le f(a)$, apply what we have just proved to $-f$, continuous by part (1) of Theorem 3.3, and to the value $-c$: this produces a point x with $-f(x) = -c$, that is, $f(x) = c$. □

From that, we obtain our first *fixed point theorem* in $\mathbb{R}$ for free.

Theorem 3.8 (Brouwer in $\mathbb{R}$)**.** *Let $a \le b$. Every continuous function $f : [a, b] \to [a, b]$ has a fixed point: there is $x^* \in [a, b]$ such that $f(x^*) = x^*$.*

Proof. Let $g(x) = f(x) - x$, continuous on $[a, b]$ by part (1) of Theorem 3.3. Since f takes its values in $[a, b]$, we have $f(a) \ge a$ and $f(b) \le b$, so

$$g(a) = f(a) - a \ge 0, \qquad g(b) = f(b) - b \le 0.$$

Hence $g(b) \le 0 \le g(a)$, and Theorem 3.7 applied to g with $c = 0$ gives $x^* \in [a, b]$ with $g(x^*) = 0$, which means $f(x^*) = x^*$. □