Sets, Functions, Logic, and Proofs

Lecture 1

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Econ 8000 — Math Camp

Sets

Sets and membership

A *set* is a collection of objects which we call *elements*.

We typically denote sets by capital letters and elements by lowercase letters.

If x is an element of a set A, we write $x \in A$.

If x is not an element of A, we write $x \notin A$.

- Example: If $A = \{1, 2, 3\}$, then $1 \in A$ and $4 \notin A$.

Describing a set

There are multiple ways to describe a set.

- One is to list its elements, as in

$$A = \{1, 2, 3\}.$$

- Another is to describe a property that characterizes its elements, as in

$$B = \{x \in \mathbb{R} : x > 0\},$$

  which is the set of positive real numbers.

Some examples of sets

- The empty set $\emptyset$.
- The set of integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
- The set of real numbers $\mathbb{R}$.
- The set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- The set of rational numbers $\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\}\}$.
- The set of all subsets of a finite set S, called the power set and denoted 2^S.

Inclusion and equality

Definition 1.1 (Inclusion and equality)

Write $A \subseteq B$ (A is a *subset* of B) if every element of A belongs to B.

The sets are equal if $A \subseteq B$ and $B \subseteq A$.

Write $A \subset B$ (A is a *proper subset* of B) if $A \subseteq B$ and $A \neq B$.

- Example: The statements

$$\{1,2\} \subseteq \{1,2,3\}, \qquad \{1,2,3\} = \{3,2,1\}, \qquad \mathbb{Z} \subset \mathbb{R}, \qquad \mathbb{N} \subset \mathbb{Z}$$

  are all true.
- Furthermore, for any set S, we have $\emptyset \subseteq S$ and $S \subseteq S$.

Set operations

Definition 1.2 (Set operations)

For sets A and B, the *union*, *intersection*, and *set difference* are

$$A \cup B = \{x : x \in A \text{ or } x \in B\},$$

$$A \cap B = \{x : x \in A \text{ and } x \in B\},$$

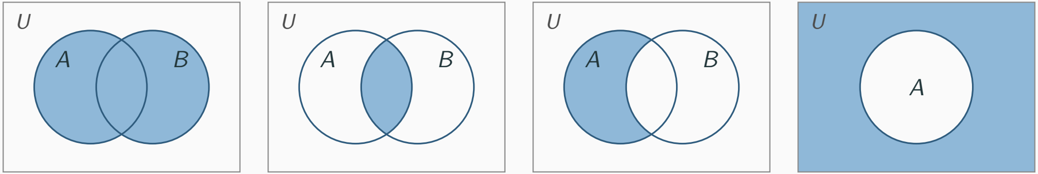
$$A \setminus B = \{x \in A : x \notin B\}.$$

The *complement* of a set A relative to a universal set U is

$$A^c = \{x : x \in U \text{ and } x \notin A\} = U \setminus A.$$

Venn diagrams

These operations are typically illustrated by Venn diagrams.



$A \cup B$ $\quad$ $A \cap B$ $\quad$ $A \setminus B$ $\quad$ $A^c = U \setminus A$

Figure 1.1

Unions and intersections of more than two sets

Given a collection of sets $\{A_i\}_{i \in I}$ indexed by a set I, we define

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}, \qquad \bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for every } i \in I\}.$$

Set algebra

Theorem 1.3 (Set algebra)

For sets A, B, and C,

1. $A \cup B = B \cup A$, $A \cap B = B \cap A$ *(commutativity),*
2. $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$ *(associativity),*
3. $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$ *(absorption),*
4. $A \setminus B = A \cap B^c$ *(set difference),*
5. $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$ *(De Morgan's laws).*

- The proof is left as an exercise.

Disjoint sets and partitions

We say that sets are *disjoint* if their intersection is empty.

- Example: The sets of even and odd integers are disjoint.

A *partition* of S is a collection of nonempty, pairwise disjoint subsets whose union is S.

- Example: The sets of even and odd integers form a partition of $\mathbb{Z}$.

Cartesian product

Definition 1.4

A *Cartesian product* of two sets A and B is the set

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

- Example: If $A = \{1, 2\}$ and $B = \{x, y\}$, then

$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}.$$

Finite products and $\mathbb{R}^n$

The product $A_1 \times \cdots \times A_n$ consists of tuples $(a_1, \ldots, a_n)$ satisfying $a_i \in A_i$ for every i.

An n-fold product of a set A with itself is denoted $A^n = A \times \cdots \times A$.

The most commonly used in economics example of an n-fold product is $\mathbb{R}^n$.

- In this course, we treat elements of $\mathbb{R}^n$ as column vectors:

  $$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n,$$

  but that convention is not important until we discuss linear algebra.

Coordinatewise comparisons

Sometimes we need to compare vectors in $\mathbb{R}^n$.

- While not all vectors are comparable, we can compare them coordinatewise.

Definition 1.5 (Coordinatewise comparisons)

For $x, y \in \mathbb{R}^n$, write

$$x \geq y \text{ if } x_i \geq y_i \text{ for every } i,$$

$$x > y \text{ if } x \geq y \text{ and } x \neq y,$$

$$x \gg y \text{ if } x_i > y_i \text{ for every } i.$$

The nonnegative and positive orthants are

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}, \qquad \mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x \gg 0\}.$$

Functions

Functions

Definition 2.1 (Function)

Let X and Y be two sets.

A *function* $f : X \to Y$ is a rule that assigns to each $x \in X$ exactly one $f(x) \in Y$.

We also call f a *map*, or a *mapping*, from X to Y.

Domain, codomain, image, and range

For a function $f : X \to Y$, we call X the *domain* and Y the *codomain*.

For a subset $A \subseteq X$ of the domain, the *image* of A is the set

$$f(A) = \{f(x) : x \in A\} \subseteq Y$$

of values taken by f on A.

- In particular, we call $f(X)$ the *range* of f.

Preimage and graph

The *preimage* of a subset $B \subseteq Y$ of the codomain is the set

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \subseteq X$$

of points in the domain that map into B.

The *graph* of f is the set of pairs

$$\{(x, f(x)) : x \in X\} \subseteq X \times Y.$$

Definition 2.2 (Injective, surjective, bijective)

A function $f : X \to Y$ is

1. *injective*, or *one-to-one*, if $f(x) = f(x')$ implies $x = x'$ for all $x, x' \in X$;
2. *surjective*, or *onto*, if its range is all of Y, that is, $f(X) = Y$;
3. *bijective* if it is both injective and surjective.

- Injectivity says that distinct points of the domain have distinct values.
- Surjectivity says that every $y \in Y$ equals $f(x)$ for at least one $x \in X$.

Surjectivity depends on the codomain

Whether a function is surjective depends on the declared domain and codomain.

- Example: $f(x) = x^2$ is not surjective if $f : \mathbb{R} \to \mathbb{R}$, but is if $f : \mathbb{R} \to \mathbb{R}_+$.
- Example: $g(x) = x^3$ is surjective if $g : \mathbb{R} \to \mathbb{R}$ or $g : \mathbb{R}_+ \to \mathbb{R}_+$.

Composition

Definition 2.3 (Composition)

Let $f : X \to Y$ and $g : Y \to Z$, so that the codomain of f is the domain of g.

The *composition* of f and g is the function $g \circ f : X \to Z$ defined by

$$(g \circ f)(x) = g(f(x)) \quad \text{for every } x \in X.$$

The preimage of a point

Let $f : X \to Y$ be a function and let $y \in Y$.

The set $f^{-1}(\{y\})$ contains all elements of X that map to the point y.

In general, this set may be empty, contain one point, or contain multiple points.

- If f is surjective, it contains at least one point for every $y \in Y$.
- If f is injective, it contains at most one point.
- If f is bijective, it contains exactly one point.

Inverse

Abusing notation slightly, we write $f^{-1}(y)$ for the unique point of X that maps to y.

- That is, $f^{-1}(\{y\}) = \{f^{-1}(y)\}$.

Definition 2.4 (Inverse)

Let $f : X \to Y$ be a bijective function.

The *inverse* of f is the function $f^{-1} : Y \to X$.

It assigns to each $y \in Y$ the unique $x \in X$ with $f(x) = y$.

f and f^{-1} undo each other

The function f and its inverse f^{-1} undo each other, in the sense that

$$f^{-1}(f(x)) = x \;\text{ for every } x \in X, \qquad f(f^{-1}(y)) = y \;\text{ for every } y \in Y.$$

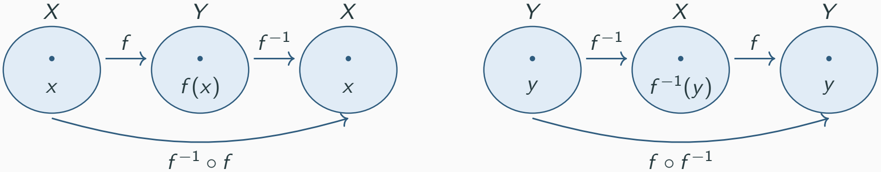


Figure 2.1

Existence of an inverse

If we tried to define an "inverse" for a function that is not bijective, we would fail.

Theorem 2.5 (Existence of an inverse)

For a function $f : X \to Y$, the following are equivalent.

1. *f is bijective.*
2. *There is a function $g : Y \to X$ with*

$$(g \circ f)(x) = x \text{ for every } x \in X, \qquad (f \circ g)(y) = y \text{ for every } y \in Y.$$

Moreover, the function g in (2) is unique and equals f^{-1}.

Logic

Statements

A *statement* is a sentence that is either true or false.

- Example: "$2 + 2 = 4$" is true and "$2 + 2 = 5$" is false.

Negation, conjunction, and disjunction

The *negation* $\neg P$ is true if and only if P is false.

The *conjunction* $P \wedge Q$ is true if and only if both P and Q are true.

The *disjunction* $P \vee Q$ is true if and only if at least one of P or Q is true.

We typically illustrate these logical operations with truth tables:

P	$\neg P$
T	F
F	T

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Implication

Perhaps the most important logical operation is implication.

- We define it via a truth table as follows:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Reading an implication

The implication $P \Rightarrow Q$ is read

- "P implies Q,"
- "if P, then Q,"
- "P is sufficient for Q,"
- "Q is necessary for P."

Converse, contrapositive, and equivalence

The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$.

Its contrapositive is $\neg Q \Rightarrow \neg P$.

If $P \Rightarrow Q$ and $Q \Rightarrow P$ both hold, we write $P \Leftrightarrow Q$.

- We say that P and Q are *equivalent*, or P if and only if (IFF) Q.

Contrapositive law

Theorem 3.1 (Contrapositive law)

$$(P \Rightarrow Q) \iff (\neg Q \Rightarrow \neg P).$$

Contrapositive law

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$$(P \Rightarrow Q) \iff (\neg Q \Rightarrow \neg P).$$

Proof.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$\neg Q$	$\neg P$	$\neg Q \Rightarrow \neg P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

$\square$

An implication $P \Rightarrow Q$ and its converse $Q \Rightarrow P$ are not equivalent.

Theorem 3.2 (Negation rules for statements)

For statements P and Q,

$$\begin{aligned} \neg(P \wedge Q) &\iff \neg P \vee \neg Q, \\ \neg(P \vee Q) &\iff \neg P \wedge \neg Q, \\ \neg(P \Rightarrow Q) &\iff P \wedge \neg Q. \end{aligned}$$

Properties

A *property* on a set X is a sentence $P(x)$ that is true or false once x is specified.

- Example: If $X = \mathbb{Z}$ and $P(x)$ is "x is even," then $P(2)$ is true and $P(3)$ is false.

Quantifiers

To turn a property into a statement, we use quantifiers.

- The *universal quantifier* $\forall$ means "for every."
- The *existential quantifier* $\exists$ means "there exists."

Example: Let $X = \mathbb{Z}$ and let $P(x)$ be "x is even."

- $\forall x \in \mathbb{Z},\ P(x)$ is false, since 3 is not even.
- $\exists x \in \mathbb{Z},\ P(x)$ is true, since 2 is even.

Negation rules for quantifiers

Theorem 3.3 (Negation rules for quantifiers)

For a property $P(x)$ on a set X,

$$\neg(\forall x \in X,\ P(x)) \iff \exists x \in X,\ \neg P(x),$$
$$\neg(\exists x \in X,\ P(x)) \iff \forall x \in X,\ \neg P(x).$$

Example: $A \subseteq B$ means $\forall x\,(x \in A \Rightarrow x \in B)$.

- Its negation is

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$$\neg(\exists x \in X,\ P(x)) \iff \forall x \in X,\ \neg P(x).$$

Example: $A \subseteq B$ means $\forall x\,(x \in A \Rightarrow x \in B)$.

- Its negation is

$$\exists x\,(x \in A \text{ and } x \notin B).$$

Proofs

What a proof is

A *theorem* is a statement that has been shown to be true.

A *proof* is the argument that shows it.

Hammack's *Book of Proof* describes a proof as follows:

> *"A proof of a theorem is a written verification that shows that the theorem is definitely and unequivocally true. A proof should be understandable and convincing to anyone who has the requisite background and knowledge."*

Writing a proof

A proof is typically written in ordinary prose plus mathematical notation.

However, every step must be justified by something already accepted:

- a hypothesis of the theorem,
- a definition,
- a result established earlier,
- or a rule of logic.

Proofs are often accompanied by pictures.

A picture usually does not replace a formal argument.

Direct proof

In a *direct proof* of an implication $P \Rightarrow Q$, we assume P and derive Q.

A direct proof

Lemma 4.1

If n is an even integer, then $-5n-3$ is an odd integer.

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Proof.

Let n be an arbitrary even integer.

Then $n = 2a$ for some $a \in \mathbb{Z}$, so

$$-5n - 3 = -5(2a) - 3 = -10a - 3 = 2(-5a - 2) + 1.$$

Now set $b = -5a - 2$.

Since a is an integer, so is b, and we obtain $-5n - 3 = 2b + 1$.

Hence $-5n - 3$ is odd. $\square$

Proof by contrapositive

A *proof by contrapositive* of $P \Rightarrow Q$ is a direct proof of $\neg Q \Rightarrow \neg P$.

- We assume $\neg Q$ and derive $\neg P$.

It is worth switching when the negations are the easier statements to work with.

The lemma again, by contrapositive

Lemma 4.1

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The lemma again, by contrapositive

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If n is an even integer, then $-5n-3$ is an odd integer.

Second proof.

We prove the contrapositive: if $-5n-3$ is even, then n is odd.

Let n be an integer for which $-5n-3$ is even, so that $-5n-3=2b$ for some $b \in \mathbb{Z}$.

Adding $6n+3$ to both sides gives

$$n = (-5n-3) + 6n + 3 = 2b + 6n + 3 = 2(b+3n+1) + 1.$$

Now set $c = b+3n+1$, which is an integer, so that $n = 2c+1$.

So n is odd, and therefore not even. $\square$

Proof by contradiction

To prove a statement S, we assume that S is false.

- We derive some statement C and its negation $\neg C$.
- But $C \wedge \neg C$ is false whatever C is.
- So the assumption fails and S is true.

The method is also called *reductio ad absurdum*.

It is justified by the equivalence

$$S \iff \big(\neg S \Rightarrow (C \wedge \neg C)\big),$$

which holds for every statement C.

In practice

Write the negation down first.

- For $P \Rightarrow Q$ the negation is $P \wedge \neg Q$.

You will not know which C ends the proof.

- Derive consequences and watch for a statement and its negation.
- Usually you contradict something you assumed.
- Sometimes you contradict a fact you already know.

A proof by contradiction

Lemma 4.2

Let $x \in \mathbb{R}$. If $x \leq \varepsilon$ for every real number $\varepsilon > 0$, then $x \leq 0$.

A proof by contradiction

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Let $x \in \mathbb{R}$. If $x \leq \varepsilon$ for every real number $\varepsilon > 0$, then $x \leq 0$.

Proof.

To assume the implication false is to assume $x \leq \varepsilon$ for every $\varepsilon > 0$ and $x > 0$.

By the second assumption, $x/2 > 0$.

The first holds for every $\varepsilon > 0$, so it holds for $\varepsilon = x/2$, giving $x \leq x/2$.

Subtracting $x/2$ from both sides and multiplying by 2 gives $x \leq 0$.

The statement C is $x \leq 0$, and its negation $x > 0$ is one of our assumptions.

So C is both true and false, a contradiction. $\square$

Principle of mathematical induction

A claim indexed by $\mathbb{N}$ is reduced by the theorem below to two checks.

Theorem 4.3 (Principle of mathematical induction)

For each $n \in \mathbb{N}$, let $P(n)$ be a statement. Suppose that

1. *$P(1)$ is true, and*
2. *for every $n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$.*

Then $P(n)$ is true for every $n \in \mathbb{N}$.

- Condition (1) is the *base step*.
- Condition (2) is the *induction step*.
- Its assumption $P(n)$ is the *induction hypothesis*.

Lemma 4.4 (Bernoulli's inequality)

Let $x \in \mathbb{R}$ with $x > -1$. Then $(1+x)^n \geq 1 + nx$ for every $n \in \mathbb{N}$.

Bernoulli's inequality

Lemma 4.4 (Bernoulli's inequality)

Let $x \in \mathbb{R}$ with $x > -1$. Then $(1+x)^n \geq 1 + nx$ for every $n \in \mathbb{N}$.

Proof.

Fix $x > -1$ and let $P(n)$ be the inequality.

Base step. For $n = 1$ both sides equal $1 + x$, so $P(1)$ holds.

Induction step. Let $n \in \mathbb{N}$ be arbitrary and assume $(1+x)^n \geq 1 + nx$.

Multiply the induction hypothesis by $1 + x$, which is positive because $x > -1$:

$$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x) = 1 + (n+1)x + nx^2.$$

Since $nx^2 \geq 0$, we have $(1+x)^{n+1} \geq 1 + (n+1)x$, which is $P(n+1)$. $\square$